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Semi-Lagrangian finite volume transport scheme on adaptively refined grids

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Abstract: This paper describes a simple, stable and accurate finite volume transport scheme for convection-dominated flow simulations on grids featuring hanging nodes as a result of h-refinement. The method's novelty is that it employs finite volume formulation to compute face fluxes of an Eulerian cell using the transport variable interpolated at its originating location obtained by Lagrangian formulation. While higher order interpolation schemes can yield better accuracy, this study uses bilinear interpolation to illustrate that the devised method is stable and preserves the shape of discontinuous fields. However, it yields non-monotonic variation which can be removed by limiters or reconstruction methods. Case studies, i.e. linear translation, rotational translation, deformation under pure shear and large deformation under a reversing vortex field, demonstrate scheme's potential for interfacial flow simulations using level-set method.

Keywords: Advection scheme, hanging node, finite volume method.

1 Introduction

Many flow problems tackle with the advection of a scalar, ϕ, as defined below:

$$\frac{\partial \phi}{\partial t} + u\,\nabla\phi = 0 \tag{1}$$

where ϕ can have discontinuities/sharp variations in space. There exists large number of techniques for finite volume, finite element and finite difference methods with a goal to obtain stable, non-dissipative solutions of the problem as defined in Eq. (1). Finite volume methods utilize integral form of Eq. (1) as given in Eq. (2).

$$\frac{\partial}{\partial t}\int_{cv} \phi dV + \oint_{cs} \phi\,(\vec{\mathrm{u}}\cdot\hat{\mathrm{n}})dA = 0 \tag{2}$$

The first term in Eq. (2) is typically replaced by cell averages while the second term requires evaluation of fluxes at the control surfaces, which can lead to numerical instabilities when a bias based on transport direction is not considered. Possible common methods that avoid such difficulties include donor-cell advection (or first order upwind), using piecewise linear interpolation, and slope/flux limiters. Among these, higher order methods require evaluation of nearby gradients for flux computation and bring cells other than those around the relevant face into the scheme. This brings algorithmic difficulties especially in multi-dimensional problems when the underlying mesh contains hanging nodes as a result of adaptive refinement.

This study develops a simple, stable and a second order accurate scheme that is primarily suitable for locally refined curvilinear grids which may include cells with hanging nodes. The accuracy of the proposed algorithm can be controlled by h-refinement rather than p-refinement.

2 Numerical method

Semi-Lagrangian methods utilize the exact solution of Eq. (1), which is given in Eq. (3), in flux calculations.

$$\phi^{n+1}(\vec{x}) = \phi^{n}(\vec{x} - \vec{u}t) \tag{3}$$

The flux at a given face is defined as follows:

$$F_f = \phi_f(\vec{u}_f \cdot \hat{n}_f)A_f \tag{4}$$

As the flux changes in time due to the change in ϕ, $\vec{u}$ and $\hat{n}$ for an arbitrary flow field, total flux can be approximated as follows:

$$\Phi_f\Big|_n^{n+1} = \int_{t^n}^{t^{n+1}} F_f \, dt \approx w_1 F^n + w_2 F^{n+1/2} + w_3 F^{n+1} \tag{5}$$

where w_1, w_2 and w_3 are interpolation weights. It is a common approach to use $w_1 = w_3 = 0$ and $w_2 = 1$ to find higher order approximations. For instance, when the flux is computed in one-dimensional space with a linear approximation for $\phi^{n+1/2}$ at $\vec{x}_f - \vec{u}\Delta t/2$ using ϕ^n of the current time step, it yields a second order Lax-Wendroff method [1].

$$\phi_P^{n+1} = \phi_P^n - \frac{\lambda}{2}(\phi_E^n - \phi_W^n) + \frac{\lambda^2}{2}(\phi_E^n - 2\phi_P^n + \phi_W^n) \tag{6}$$

where λ is the CFL number and defined as $u\Delta t/\Delta x$. In this study, weights are selected considering half weights at the next and current time step; i.e. $w_1 = w_3 = 0.5$ and $w_2 = 0$; along with a second order Explicit Runge Kutta method.

For higher dimensions, different interpolation schemes can be utilized to account for the direction of the velocity by shifting the vertices to a new location $\vec{x}_v^b = \vec{x}_v^a - \vec{u}_v^a\Delta t$ to compute the fluxes at their new locations [2–6].

This study adopts interpolation zones for each vertex, to ease estimation of any field variable at an arbitrary location. The interpolation zones are illustrated in Fig. (1) using dashed lines for a given vertex

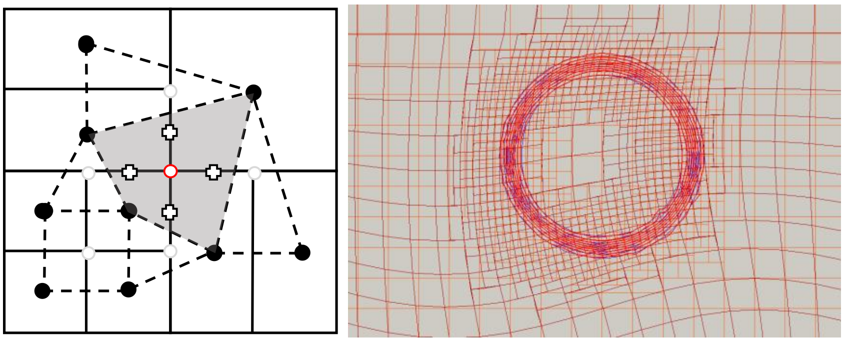

Figure 1: a) Interpolation zone governed by a vertex; b) interpolation of level contours after shifting vertices on characteristic lines

indicated by a void circle marker. Fig. (1) also indicates storage locations using filled circles at cell-centers where all variables, ϕ and $\vec{u}$; and face centers (f) are marked with void plus markers. Present scheme use bilinear interpolation method for estimating the transport variable at this arbitrary location.

3 Results

A scalar function which varies between 0 and 1 is utilized to represent various geometries. The value of the scalar function is 1 in every cell inside the geometry while it is set to zero everywhere outer domain. The shape of the geometry is determined exactly at a value of 0.5. Four case studies are considered: (i) linear translation, (ii) rotational translation, (iii) deformation under shear, and (iv) large deformation of a circular interface under a vortex-field.

Figure (2) compares the results of linear translation at $t = 0$, $t = 0.5t_f$ and $t = t_f$ for first order upwind FOU and semi-Lagrangian method (SML). It is clear that the diffusion in FOU causing the geometry to diffuse is reduced in semi-Lagrangian method.

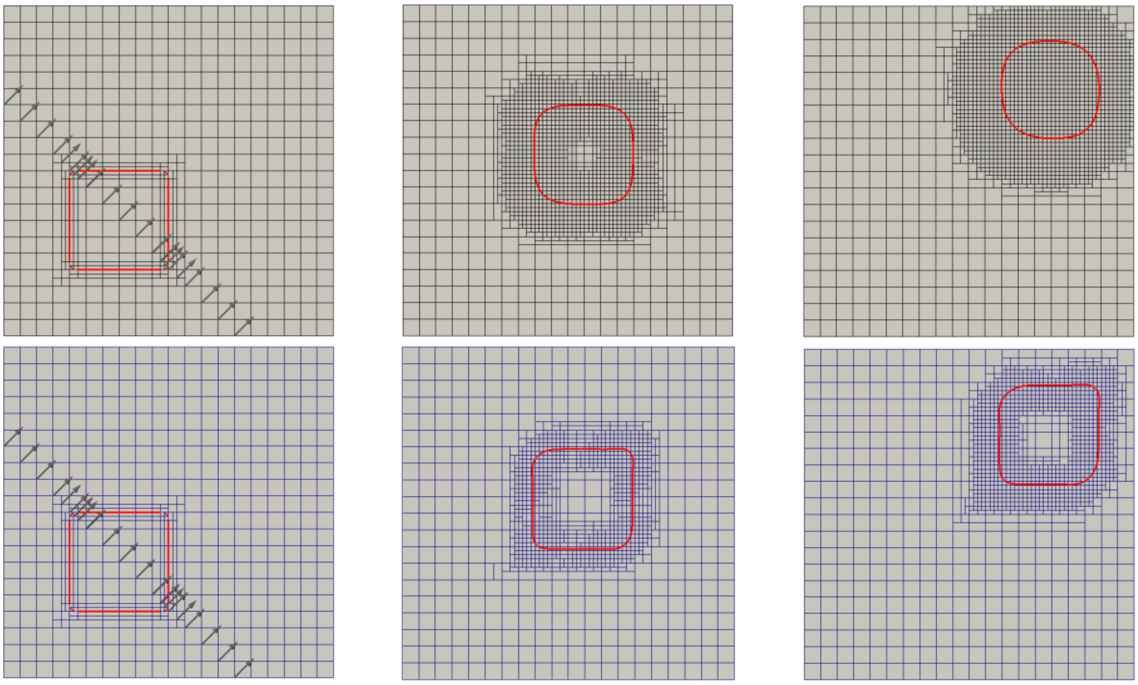

Figure 2: Linear translation. Results in the top row are obtained using FOU, while those in the bottom row are obtained using SLM

Figure (3) compares the results of linear rotation for first order upwind and semi-lagrangian method. They both lose geometric information during 90 degree rotation. Also SML is observed to produce spurious oscillation in ϕ inside the geometry causing more refined cells.

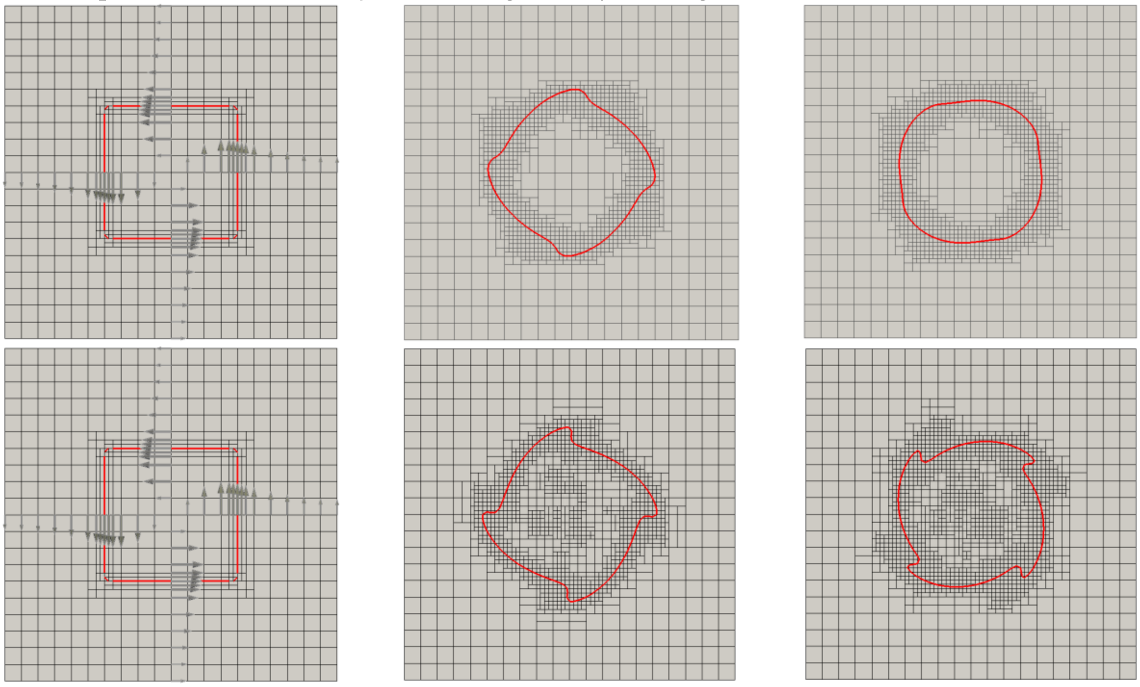

Figure 3: Linear rotation. Results in the top row are obtained using FOU, while those in the bottom row are obtained using SLM

Figure (4) compares the results of deformation under shear. FOU and SLM both produce reasonable results.

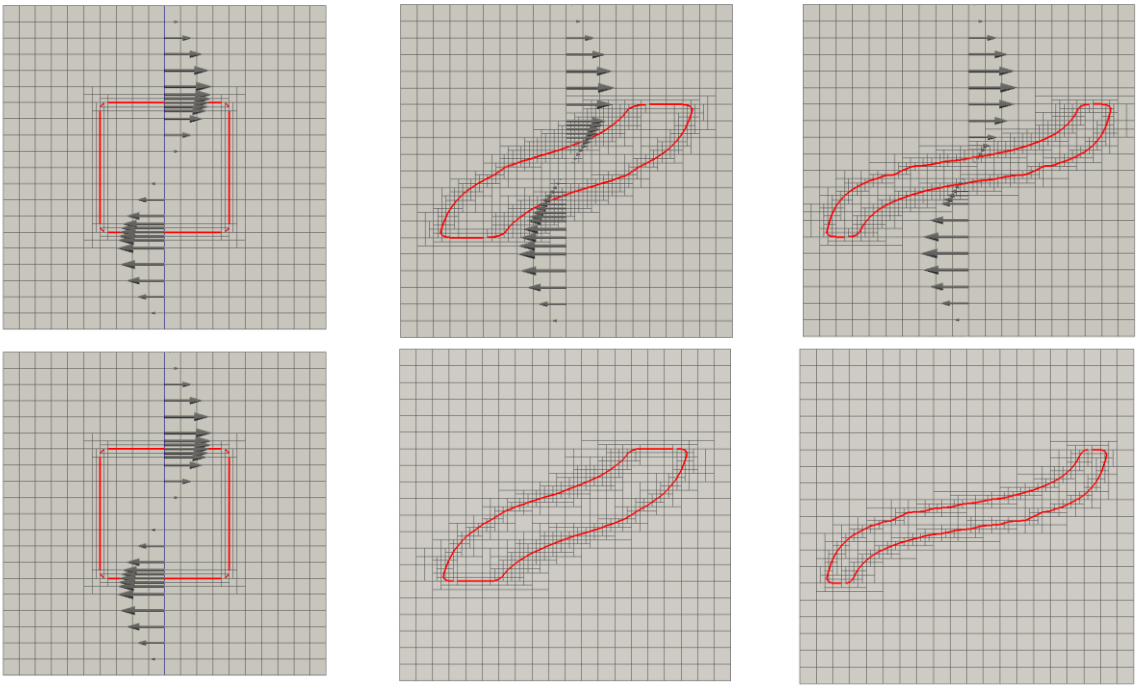

Figure 4: Deformation under shear. Results in the top row are obtained using FOU, while those in the bottom row are obtained using SLM

Figure (5) shows the large deformation case; in which the velocity field changes in time as contrast to previous cases. The velocity field in unit square is defined as follows:

$$u = u_o \sin(\pi y) \cos(\pi y) \sin^2(\pi x) \cos(\pi t / t_f) \quad (7)$$

$$v = v_o \sin(\pi x) \cos(\pi x) \sin^2(\pi y) \cos(\pi t / t_f) \quad (8)$$

A circle is placed at $x = 0.5$ and $x = 0.7$, deformed under the velocity field. As the velocity field smoothly reverses in time, the circle is expected to be recovered at the end of the simulation ($t = t_f$). As it can be seen from Fig. (5), first order upwind diffuses so fast that the contour at $\phi = 0.5$ disappears very quickly. On the other hand, SML recovers a circular shape with reasonable error. The refinement density is also observed to be much smaller in SML than FOU.

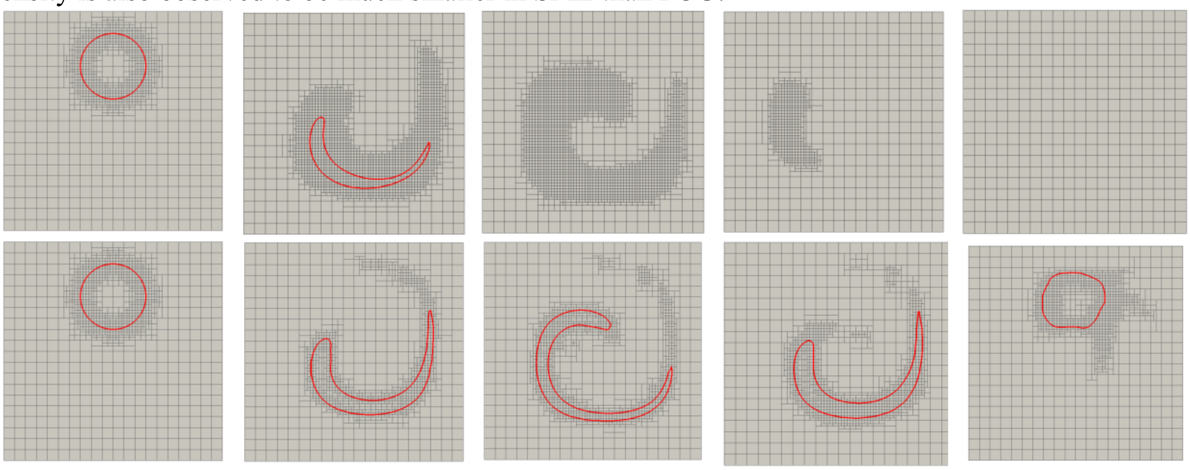

Figure 5: Large deformation of a circle in a reversing vortex field. Results in the top row are obtained using FOU, while those in the bottom row are obtained using SLM

4 Conclusion and Future Work

In this short paper, semi-Lagrangian advection scheme is investigated for its use in adaptively refined grids with hanging nodes. The scheme is found to be promising as it is simple to implement. Further improvement on the order of the interpolation scheme is expected to increase method's order of accuracy.

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