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SIMULATION OF NAVIER-STOKES FLOWS INCLUDING RIGID PARTICLES

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Abstract: The present paper proposes a numerical approach to simulate the motion of rigid particles in an incomprssible fluid. The rigid motion is enforced by penalizing the strain tensor on the rigid domain. The method is based on a variational formulation on the whole domain (fluid and solid). We implemented this method and we simulated several test cases. Finaly to validate this approach, we apply this method to simulate the case of a particle which is subjected to shear fields.

Keywords: Penalisation Method, Rigid particle, Navier-Stokes, Sheared particle.

1 Introduction

Theoretical approaches aiming at describing the behavior of suspensions of rigid or deformable particles have been limited for a long time to dilute suspensions. This approach has begun with the seminal work of Einstein [1, 2] in 1906 and 1911 on the effective viscosity of a dilute suspension of rigid spheres and, since then, it has been completed by numerous works [3, 4, 5, 6, 7, 8] related to the framework of dilute suspensions. From the prediction point of view, most of these works deal with weak solid concentrations and small solid sizes in which pairwise interactions lead to the behavior of the suspensions.

When dealing with a big rigid particles, a new approach is required. Several strategies have been proposed in the last two decades to simulate the motion of rigid bodies in a viscous fluid. A first class of methods relies on methods with a mesh in the fluid domain, by computing the flow in the fluid domain (which is complex because of the inclusions). Then, it is possible to compute the forces exerted on the particles and, as a consequence, the velocity perturbations. This methods are relies on a moving mesh following the fluid domain [9, 10, 11, 12]. The second class is the fictitious domain methods also called domain embedding methods: the idea is to extend a problem defined on a time-dependent, complex domain (the fluid domain) to a larger one (fixed) called the fictitious domain [13, 14]. Penalty methods are another class of fictitious domain strategies [15].

The penalty method is based on a reformulation of the stress tensor for canceling the deformation rate in the volume occupied by the particle. It consists on constraining the movement of the fluid to be a rigid body motion identical to that of a particle by locally increasing the viscosity of the fluid [16, 17, 18]. Recently this method has been extended to manage of stress rigid motion for a particle in a fluid to an approach of finite difference type and then by a finite elements method. [19, 20]. This method is implemented on a general Finite Element solver which we use to make numerical tests.

In this paper we present a method of simulating the movement of one or more convex rigid body in a Newtonian incompressible fluid. We used a penalty method which is based on a reformulation of the stress tensor which allows the canceling of the deformation rate in the volume occupied by the particle. The objective is to develop a code from FreeFem ++ that simulates Stokes or Navier-Stokes flows (with low Reynolds number) in the presence of solid particles. To validate this method, we apply this method to simulate the case of a particle which is subjected to shear fields.

2 Modelling rigid body flows

We consider a connected, bounded and regular domain $\Omega \subset \mathbb{R}^2$ (see Fig.1) and we denote by $(B_i)_{i=1,\cdots,N}$ the rigid particles, strongly included in Ω. B denotes the whole rigid domain: $B = \cup_i B_i$. The domain $\Omega \setminus \overline{B}$ is filled with Newtonian fluid governed by the Navier-Stokes equations. We note μ the viscosity of the fluid, p the presure and $\mathrm{f_f}$ the external forces exerted on it. Since we consider a Newtonian fluid, the stress tensor $\underline{\underline{\sigma}}$ is given by the following relation (see Eq. (1)):

$$\underline{\underline{\sigma}} = 2\mu\mathbb{D}(\mathbf{u}) - p\mathbb{I}, \quad \text{where} \quad \mathbb{D}(\mathbf{u}) = \frac{\nabla(\mathbf{u}) + (\nabla(\mathbf{u}))^T}{2} \tag{1}$$

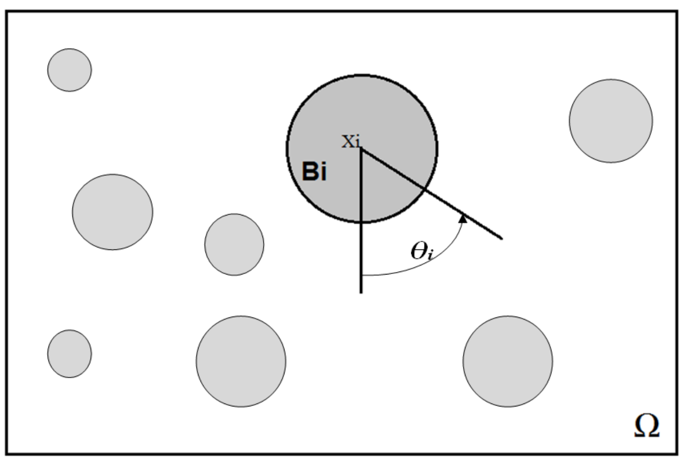


Figure 1: Partcicles B_i in a Newtonian fluid.

We consider homogeneous Dirichlet conditions on $\partial\Omega$. The presence of viscosity imposes a no-slip condition on the boundary ∂B of the rigid domain.
At the initial time the particles with density ρ_i are distributed randomly over the fluid. The position of the center of the ith particle is denoted by x_i, by v_i and ω_i its translational and angular velocities. We denote by m_i and J_i the mass and the kinematic momentum about its center of mass:

$$m_i = \int_{B_i} \rho_i, \quad J_i = \int_{B_i} \rho_i \|x - x_i\|^2 \tag{2}$$

We have to find the velocity $\mathrm{u}(u_1, u_2)$ and the pressure field p defined in $\Omega \setminus \overline{B}$, as well as the velocities of the particles$\mathbf{V} := (\mathrm{v_{i=1,\ldots,N}}) \in \mathbb{R}^{2N}$ and $\omega := (\omega_{\mathrm{i=1,\ldots,N}}) \in \mathbb{R}^N$ such that (see Eq. (3)):

$$\begin{cases} \rho_{\rm f}(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}) - \mathrm{div}(\underline{\sigma}) = \mathbf{f}_{\rm f} & \text{in } \Omega \setminus \overline{B}, \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in } \Omega \setminus \overline{B}, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \mathbf{u} = \mathbf{v}_i + \omega_i(x - x_i)^{\perp} \text{ on } \partial B, & \forall i \in \{1,\dots,N\} \end{cases} \tag{3}$$

where $\rho_{\rm f}$ denotes the density of the fluid and $\mathbf{f}_{\rm f} = \rho_{\rm f} g e_y$ is the external force exerted on the fluid (gravity forces). The fluid exerts hydrodynamic forces on the particles. Newton's second law for these particles is written then as follows (see Eq. (4)):

$$\begin{cases} m_i \frac{dV_i}{dt} = \int_{B_i} \mathrm{f_i} - \int_{\partial B_i} \underline{\sigma} n, \\ J_i \frac{d\omega_i}{dt} = \int_{B_i} (x - x_i)^{\perp}.\mathrm{f_i} - \int_{\partial B_i} (x - x_i)^{\perp}.\underline{\sigma} n, \end{cases} \tag{4}$$

Where, $\mathrm{f_i}$ denotes the external non-hydrodynamical forces exerted on the sphere, such as gravity : $\mathrm{f_i} = -\rho_i g e_y$.

2.1 Variational Formulation

The functionnal spaces are defined as:

$$L^2(\Omega) = \left\{ f \ : \Omega \longrightarrow \mathbb{R}; \int_{\Omega} |f|^2 < +\infty \right\} \tag{5}$$

$$L_0^2(\Omega) = \left\{ f \ \in L^2(\Omega); \int_{\Omega} f = 0 \right\} \tag{6}$$

$$H^1(\Omega) = \left\{ f \ \in L^2(\Omega); \nabla f \in L^2(\Omega) \right\} \tag{7}$$

$$H_0^1(\Omega) = \left\{ f \ \in H^1(\Omega); f = 0 \ on \ \partial\Omega \right\} \tag{8}$$

$$K_{\nabla} = \left\{ u \in H_0^1(\Omega), \nabla.u = 0. \right\} : \ divergence\ space \tag{9}$$

$$\begin{aligned} K_B &= \left\{ u \in H_0^1, \forall i, \exists (v_i, \omega_i) \in \mathbb{R}^2 \times \mathbb{R}; u = v_i(t) + \omega_i (x - x_i)^{\perp} \quad in\ B. \right\} \\ &= \left\{ u \in H_0^1, D(u) = 0 \quad in\ B \right\} : \ rigid\ mouvement\ space. \end{aligned} \tag{10}$$

K_{∇} is the space of divergence free functions on Ω and K_B is the space of functions on that do not deform B. The variational formulation obtained on the whole fluid/particle domain Ω is given here after (see Eq. (11)):

$$\begin{cases} Find\ (\mathbf{u}, p) \in \mathbf{K}_B \times \mathbf{L}_0^2(\Omega)\ suchthat \\ \int_{\Omega} \tilde{\rho} \frac{D\mathbf{u}}{Dt} \mathbf{v} + 2\mu \int_{\Omega} \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) - \int_{\Omega} p \mathrm{div}(\mathbf{v}) = \int_{\Omega} \tilde{\mathrm{f}}.\mathbf{v}, \ \forall \mathbf{v} \in \mathbf{K}_B \\ \int_{\Omega} q \mathrm{div}(\mathbf{u}) = 0, \ \forall q \in \mathbf{L}_0^2, \end{cases} \tag{11}$$

with $\tilde{\rho} := \rho_f \mathbf{1}_{\Omega\setminus\overline{B}} + \sum_{i=1}^N \rho_f \mathbf{1}_{B_i}$, $\quad \tilde{\mathrm{f}} := \mathrm{f}_f \mathbf{1}_{\Omega\setminus\overline{B}} + \sum_{i=1}^N \mathrm{f}_i \mathbf{1}_{B_i}\ and\, \mathbf{K}_B = \{\mathbf{u} \in H_0^1(\Omega) \mathbb{D}(\mathbf{u}) = 0\ in B\}$.

2.2 Method of Characteristics

The time discretization is performed using the method of characteristics [21]. We obtain the following discretized scheme: for each $n > 1$:

$$\begin{cases} \dfrac{1}{\Delta t} \displaystyle\int_{\Omega} \rho^{n+1} u^{n+1}(x)\, \overline{u}\ d\Omega + 2\mu \int_{\Omega} D\left(u^{n+1}\right) : D\left(\overline{u}\right) d\Omega - \int_{\Omega} p^{n+1}\ div\left(\overline{u}\right) d\Omega \\ \quad = \dfrac{1}{\Delta t} \displaystyle\int_{\Omega} \left(\rho^{n+1} u^{n}\right) o\left(X^{n}(x)\right) \overline{u}\ d\Omega + \int_{\Omega} f^{n+1}\ \overline{u}\ d\Omega,\ \forall\ \overline{u} \in K_{B^{n+1}} \\ \displaystyle\int_{\Omega} q\ div\left(u^{n+1}\right) = 0\ \forall q \in L^2\left(\Omega\right) \end{cases} \tag{12}$$

2.3 Penalisation Method

This method is presented in [19, 22] and consists in considering the minimization problem over a constrained domain associated to eqution (12) and relaxing the constraint by introducing a penalty term in the minimized functional. The added term is the following:

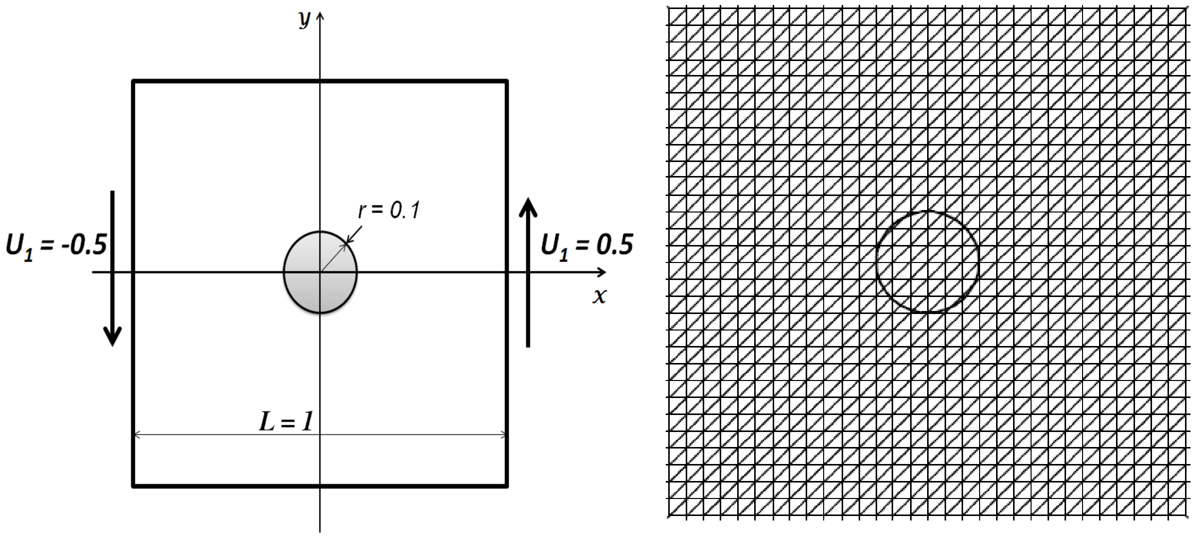
$$\frac{1}{\epsilon} = \int_{B^{n+1}} D\left(u^{n+1}\right) : D\left(u^{n+1}\right),$$

so that $D\left(u^{n+1}\right) |_{B^{n+1}} \rightarrow 0$ goes to zero when $\epsilon = 0$ goes to zero and u^{n+1} tends to be a rigid motion in B^{n+1}. Finally, there are the conditions of the rigid movement and adapted the variational formulation is obtained at the following discretization finite elements:

$$\begin{cases} Find\ u^{n+1} \in H_0^1(\Omega)\ p^{n+1} \in L^2(\Omega)\ such\ that, \\ \dfrac{1}{\Delta t} \displaystyle\int_{\Omega} \rho^{n+1} u^{n+1}.\overline{u}\ d\Omega + 2\mu \int_{\Omega} D\left(u^{n+1}\right) : D\left(\overline{u}\right) d\Omega \\ + \dfrac{2}{\epsilon} \displaystyle\int_{B^{n+1}} D\left(u^{n+1}\right) : D\left(\overline{u}\right) d\Omega - \int_{\Omega} p^{n+1}\ div\left(\overline{u}\right) d\Omega \\ \quad = \dfrac{1}{\Delta t} \displaystyle\int_{\Omega} \left(\rho^{n} u^{n}\right) o\left(\mathbf{X}^{n}.\overline{u}\right)\ d\Omega + \int_{\Omega} f^{n}\ \overline{u}\ d\Omega,\ \forall\ \overline{u} \in H_0^1(\Omega) \\ \displaystyle\int_{\Omega} q\ div\left(u^{n+1}\right) = 0,\ \forall q \in L^2\left(\Omega\right) \end{cases} \tag{13}$$

3 Sheared particle

We consider the instantaneous problem of a particle in a Newtonnain fluid. The computational domain is a square 1cm wide and a particle of radius 0.1cm is situated at its center . The right and left walls of the domain impose a shearing motion to the system, the viscosity of the fluid is equal to 1 . A cartesian meshes are used (see Fig. 2).



a) *b)*

Figure 2: Sheared particle: left: physical domain; right: cartesian meshes

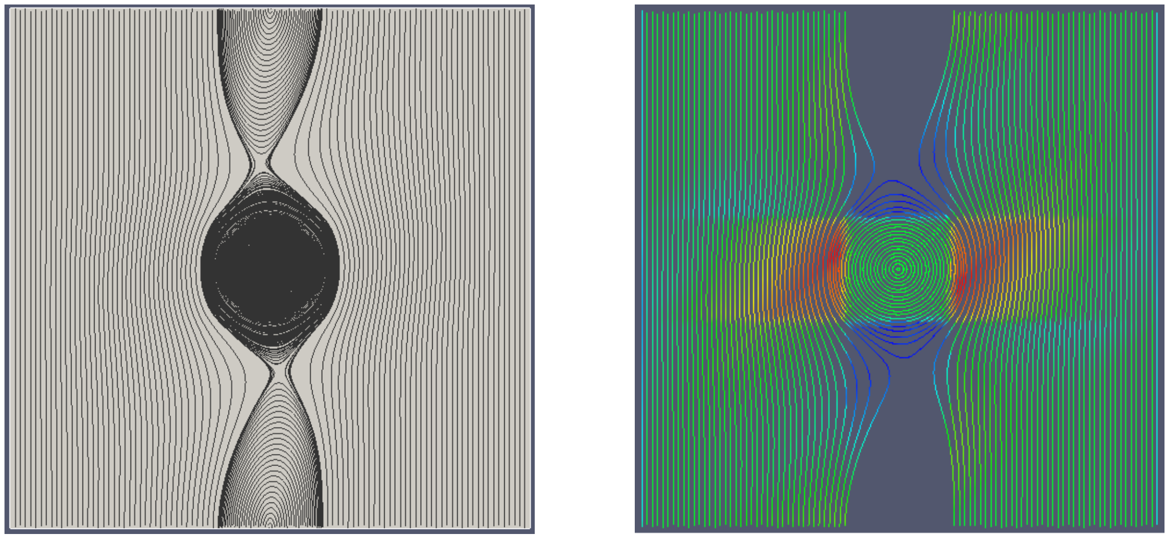
r_p	ω		r_p	ω
0.6	0.1804		0.0039	0.452976
0.45	0.249947		0.00385	0.47697
0.25	0.249492		0.00383	0.487011
0.17	0.249895		0.00381	0.497318
0.15	0.249977		0.003801	0.499937
0.1	0.251729		0.0038049	0.499989
0.01	0.356297		0.00380489	0.499995

Table 1: Angular vilocity of a sheared particle

On the table (1) below we have presented the angular velocity a function of the particle radius. We show that the angular velocity of the particule converges to the theoretical value and is equal to 0.5 [23].

$$\omega = \frac{\dot{\gamma}}{2} = \frac{2U}{2L}$$

The streamlines and the velocity field of the shear movement are respectively shown in figure 3. This figure show the streamlines of the rotational motion.



a) *b*)

Figure 3: Sheared particle: a) Stream line; b)Vilocity field

4 Conclusion

In this paper, we have proposed a strategy for the numerical modeling of the motion of a rigid particle in a Newtonian fluid. The rigid motion is imposed by penalizing the strain tensor, the time discretization is performed by using the method of characteristics.
The code was written in FreeFem++ version 3.26 and at each time step the generalized Navier-Stokes problem is solved by using standard finite elements. From the results of this case, we notice that the stress of rigid motion is taken into account. These results are similar to those existing in the literature especially those obtained by Lefebvre [20].
We verified that when the radius of the particle tends to zero we find the theoretical value of the angular velocity which converges to the theoretical value of the angular velocity $\frac{\dot{\gamma}}{2} = \frac{2U}{2L}$.

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