Numerical Modelling of Immiscible Hele-Shaw Flow with Inhomogeneous Viscosity

S.J. Jackson, H. Power and D. Giddings

Fluids and Thermal Engineering Research Group, University of Nottingham, UK

Corresponding author: henry.power@nottingham.ac.uk

Abstract: We investigate immiscible radial displacement in a Hele-Shaw cell with a temperature dependent viscosity using two coupled high resolution numerical methods. The displacement of a resident high viscosity fluid through the radial injection of a low viscosity fluid at a different temperature creates spatial gradients in the temperature field, altering the fluid viscosity and resulting interfacial displacement. The transient heat transfer is evaluated using a new auxiliary multi-zone radial basis function finite-collocation (RBF-FC) method, whilst the pressure/velocity field is solved using a coupled boundary element method (BEM) - embedded multi-zone RBF-FC method. Results reveal that the inhomogeneous viscosity accelerates the flow in hot-cold injections, creating more ramified fingering patterns and promoting the onset of viscous fingering. In cold-hot injections the interface can be significantly stabilised, delaying bifurcation. However, with a naturally unfavourable viscosity contrast between the two fluids, bifurcation will always occur eventually as the temperature field diffuses with time. We find that the thermal Peclet number has to be very small for the displacement to resemble that of corresponding isothermal cases, as large scale temperature variations still exist around the interface (and alter the perturbed velocity) even when the thermal Peclet number is relatively low.

Keywords: Thermo-Viscous fingering, Inhomogeneous Viscosity, Interface Tracking, RBF-FC, BEM.

1 Introduction

During the immiscible displacement of a high viscosity fluid by the radial injection of a low viscosity fluid, interfacial instabilities can evolve forming complex viscous fingering patterns. This process occurs in a variety of practical applications, particularly those occuring in porous media, such as enhanced oil recovery [1] and carbon sequestration [2].

One such porous media flow and the motivation behind the current work is the injection and storage of CO_2 in deep subsurface aquifers (carbon sequestration). The injection of supercritical CO_2 will typically occur at a temperature above that of the resident brine, creating temperature gradients in the domain that will alter mechanical fluid quantities such as thermal diffusivity, viscosity, density and surface tension. Understanding how temperature gradients affect the fluid properties and resulting interfacial displacement is key to understanding the CO_2 injection and storage regime. In this work, we analyse the changes in fluid viscosity due to thermal gradients in the domain and the effect this has on the immiscible interfacial displacement (often called thermo-viscous fingering) using the analogy between Hele-Shaw and porous media flows.

During thermo-viscous fingering, generally, two fronts are established in the domain: solutal and thermal. The solutal front marks the compositional transition from the injected fluid to the resident fluid. In immiscible displacement this is a sharp front controlled by capillary forces [3], where the mechanical fluid properties exhibit discontinues profiles, whereas in miscible displacement the front is smoothed as dispersion and molecular diffusion mix the fluid properties of the injected and resident fluid [4]. In miscible flows the viscosity of the fluid is determined by both the solute concentration and the temperature. However, in immiscible flows, the viscosity depends only on the fluid being considered and the temperature that the fluid exists at.

There exists a very limited number of studies on immiscible thermo-viscous fingering, with most previous research focusing on miscible cases. One notable work in the immiscible regime by Sheorey and Muralidhar finds that non-isothermal, high temperature injection of water into resident oil promotes the formation of viscous fingers and increases the sweep efficiency of the water [5].

For miscible flows, there are several previous works concerning the linear stability of the problem [4][6][7][8][9]. These analyses focus on the interaction and time-scale effects of the different scalar diffusive fronts. In the analysis of radial double-diffusive fronts, Pritchard finds that even for strongly stabilising temperature fields (i.e. they form a decreasing viscosity profile across the interface), a destabilising viscosity profile can be maintained over the front due to a strongly advective solutal regime [4], unless the thermal lag is near unity and the thermal viscosity contrast is many times higher than the viscosity contrast across the solutal front.

In this current work, we analyse the immiscible thermo-viscous fingering process in a Hele-Shaw cell due to the radial injection of a low viscosity fluid with injection temperature different to that of the high viscosity resident fluid. The work does not pretend to model the complex 3D porous media flow occurring in CO_2 injection, instead analysing the case of Hele-Shaw flow to study the effects of thermo-viscous fingering in an immiscible regime with a true multi-zone representation of the fluid flow and heat transfer.

Using a newly developed boundary element (BE) - radial basis function (RBF) method, we track the fluids' pressure, velocity and temperature through the solution of Darcy flow and the multi-zone convectiondiffusion of heat through the domain. This method allows the late stage exploration of the effects of a transient temperature field on the immiscible fluid front and the resulting displacement regime.

2 Formulation

We consider a radial Hele-Shaw cell, in which high viscosity fluid is displaced by the radial injection of a low viscosity fluid. The low viscosity invading fluid (such as CO_2) with temperature T_0 occupies the inner region Ω_1 , whilst a high viscosity fluid (such as brine) with temperature T_{∞} occupies the external region Ω_2 , shown in Figure 1. The interface separating the internal and external region has an initial perturbation given by $r = r_0 + \epsilon_0 r_0 \cos(8\theta)$, where r, r_0, ϵ_0 and θ are the interface radius, the unperturbed interface radius, the perturbation amplitude and the azimuthal angle around the interface respectively. The initial temperature $T(\boldsymbol{x})$ at a point \boldsymbol{x} is given by a smoothed step function, i.e.

$$T(\boldsymbol{x}) = T_{\infty} + 0.5(T_0 - T_{\infty}) \left(1 + \tanh\left(\frac{r_{\theta}}{\delta}\right)\right)$$
(1)

Where, r_{θ} is the radial distance from a point \boldsymbol{x} to the corresponding interface point with the same azimuthal angle. δ is the sharpness of the smoothed step function.

For the flow in a Hele-Shaw cell, the depth averaged pressure P and 2D Darcy velocity u in each fluid region l can be expressed through Darcy's law: $u_i^l(x) = -M_l(x)\frac{\partial P_l}{\partial x_i}$. The mobility M_l is related to the fluid viscosity $\mu_l(x)$ and plate spacing b by: $M_l(x) = b^2/12\mu_l(x)$. Substituting the Darcy flow expression into the conversation of mass, we find:

$$\frac{\partial}{\partial x_i} \left(M_l(\boldsymbol{x}) \frac{\partial P_l}{\partial x_i} \right) = 0 \quad \boldsymbol{x} \in \Omega_l, \quad l = 1, 2$$
⁽²⁾

At a boundary point $\boldsymbol{\xi}$ on the fluid interface S between Ω_1 and Ω_2 , there are two matching conditions (kinematic and dynamic) that must be met by the advancing interface. Firstly, the kinematic condition



Figure 1: Initial setup of the immiscible thermo-viscous fingering problem in a Hele-Shaw cell. The black fluid-fluid interface overlays a contour plot of the initial temperature field.

requires the continuity of normal velocity $u_i(\boldsymbol{\xi})n_i(\boldsymbol{\xi})$, i.e.

$$-M_1(\boldsymbol{\xi})\frac{\partial P_1(\boldsymbol{\xi})}{\partial n_i} = -M_2(\boldsymbol{\xi})\frac{\partial P_2(\boldsymbol{\xi})}{\partial n_i}$$
(3)

Secondly, the dynamic condition describes the pressure jump across the interface:

$$P_1(\boldsymbol{\xi}) - P_2(\boldsymbol{\xi}) = \frac{1}{Ca_g} \left(\frac{2}{b} + \frac{\pi}{4} k(\boldsymbol{\xi}) \right)$$
(4)

Where, Ca_g is the global capillary number and $k(\boldsymbol{\xi})$ is the curvature. The global capillary number in (4) describes the ratio of viscous driving forces to surface tension forces in terms of the characteristic mobility \hat{M}_2 and plate separation b at the origin; $Ca_g = r_0 Q/\gamma \hat{M}_2$. Here, γ is the surface tension. The displacement of the outer fluid is initiated by the injection of the inner fluid with a point source of strength Q at the origin.

In order to characterise effects coming from the non-isothermal regime, we decompose the variables that relate to the viscosity into homogeneous and perturbed components. Firstly, the viscosity of each fluid l can be represented through an exponential dependence on the temperature[4][9].

$$\mu_l(\boldsymbol{x}) = \overline{\mu_l} \exp\left[\mp \alpha_L \left(1 \pm \frac{T(\boldsymbol{x}) - T_\infty}{|T_0 - T_\infty|}\right)\right] \quad \boldsymbol{x} \in \Omega_l, \quad l = 1, 2$$
(5)

Here, the homogeneous viscosity $\overline{\mu_l}$ is that of fluid l at the injection temperature T_0 . T_{∞} is the resident fluid temperature, α_L is the rate of change of viscosity for fluid l with temperature. The minus/plus sign in $\mp \alpha_L$ is used for when the injecting fluid is hotter (+) or colder (-) than the resident fluid. This gives the

mobility of the fluid in zone l as:

$$M_l(\boldsymbol{x}) = \frac{b^2}{12\mu_l(\boldsymbol{x})} = \overline{M}_l + \tilde{M}_l(\boldsymbol{x})$$
(6)

$$\overline{M}_{l} = \frac{b^{2}}{12\overline{\mu_{l}}}, \quad \tilde{M}_{l}(\boldsymbol{x}) = \frac{b^{2}}{12\mu_{l}(\boldsymbol{x})} - \overline{M}_{l}$$

$$\tag{7}$$

We now introduce non-dimensional variables utilising the characteristic length, time, velocity, pressure and temperature of the problem, given by:

$$(\boldsymbol{x}, r, b) = r_0 (\boldsymbol{x}', r', b')$$
 $t = \frac{r_0^2}{Q} t'$ (8)

$$u_{l} = \frac{Q}{r_{0}}u_{l}' \quad l = 1, 2 \qquad P_{l} = \frac{Q}{\overline{M}_{2}}P_{l}' \quad l = 1, 2$$
(9)

$$T_l = |T_0 - T_\infty| T'_l + T_\infty \qquad D_l = QD'_l \quad l = 1,2$$
(10)

In equalities (8) - (10), apostrophes identify non-dimensional variables with t as time and \overline{M}_2 as the homogeneous mobility of the displaced fluid (corresponding to the homogeneous viscosity $\overline{\mu_2}$). D_l is the thermal diffusivity of fluid l.

The characteristic mobility and plate spacing given in the global capillary number are equal to the homogeneous values at the origin. From this point onwards, the apostrophe of all non-dimensional variables will be dropped, and all variables will be assumed to be in their non-dimensional form unless otherwise stated. We start by representing the pressure as a sum of homogeneous and perturbed components:

$$P_l(\boldsymbol{x}) = \bar{p}_l(\boldsymbol{x}) + \tilde{p}_l(\boldsymbol{x}) \tag{11}$$

Expanding (2), noting that $\partial \overline{M}_l / \partial x_i = 0$, the following equation is obtained:

$$M_l(\boldsymbol{x})\frac{\partial^2 \bar{p}_l}{\partial x_i^2} + M_l(\boldsymbol{x})\frac{\partial^2 \tilde{p}_l(\boldsymbol{x})}{\partial x_i^2} + \frac{\partial \tilde{M}_l(\boldsymbol{x})}{\partial x_i}\frac{\partial \tilde{p}_l(\boldsymbol{x})}{\partial x_i} = -\frac{\partial \tilde{M}_l(\boldsymbol{x})}{\partial x_i}\frac{\partial \bar{p}_l(\boldsymbol{x})}{\partial x_i}$$
(12)

The homogeneous pressure is constrained to satisfy Laplace's equation, in order to characterise the case of isothermal displacement, i.e $\nabla^2 \bar{p}_l = 0$. The homogeneous pressure is then subject to the following matching and asymptotic conditions:

$$\bar{p}_1(\boldsymbol{\xi}) - \bar{p}_2(\boldsymbol{\xi}) = \frac{1}{Ca_g} \left(\frac{2}{b} + \frac{\pi}{4} k(\boldsymbol{\xi}) \right) \quad for \quad \boldsymbol{\xi} \in S$$
(13)

$$\overline{M}_1(\boldsymbol{\xi})\frac{\partial \overline{p}_1(\boldsymbol{\xi})}{\partial n_i} = \overline{M}_2(\boldsymbol{\xi})\frac{\partial \overline{p}_2(\boldsymbol{\xi})}{\partial n_i} \quad for \quad \boldsymbol{\xi} \in S$$
(14)

$$\bar{p}_2(\boldsymbol{x}) \to -\frac{1}{2\pi} \ln(r) \tag{15}$$

The perturbed pressure equation (12) is then subject to the following matching conditions at the interface, S, and asymptotic conditions near the origin and in the far field:

$$\tilde{p}_1(\boldsymbol{\xi}) - \tilde{p}_2(\boldsymbol{\xi}) = 0 \quad for \quad \boldsymbol{\xi} \in S$$
(16)

$$M_1(\boldsymbol{\xi})\frac{\partial \tilde{p}_1(\boldsymbol{\xi})}{\partial n_i} = M_2(\boldsymbol{\xi})\frac{\partial \tilde{p}_2(\boldsymbol{\xi})}{\partial n_i} + \left(\beta \tilde{M}_2(\boldsymbol{\xi}) - \tilde{M}_1(\boldsymbol{\xi})\right)\frac{\partial \bar{p}_1(\boldsymbol{\xi})}{\partial n_i} \quad for \quad \boldsymbol{\xi} \in S$$
(17)

$$\frac{\partial \tilde{p}_1(\boldsymbol{x})}{\partial r} \to 0 \quad \begin{array}{c} \tilde{p}_2(\boldsymbol{x}) \to 0\\ \boldsymbol{x} \to \infty \end{array} \tag{18}$$

The obtained values of \bar{p}_l and \tilde{p}_l can be used to reconstruct the normal velocity at an interface point, $\boldsymbol{\xi}$, in

order to track the fluid:

$$U_n(\boldsymbol{\xi}) = \bar{u}_n(\boldsymbol{\xi}) + \tilde{u}_n(\boldsymbol{\xi}) \tag{19}$$

To evaluate the flow and pressure in the domain and calculate the interfacial velocity with equation (19) we require the temperature throughout in order to calculate the viscosity and its spatial derivatives. To track the temperature evolution through the domain to evaluate the viscosity (5) and resulting pressure/flow field, we solve the multi-zone convection diffusion heat equation:

$$\frac{\partial T_l(\boldsymbol{x},t)}{\partial t} + u_i(\boldsymbol{x},t)\frac{\partial T_l(\boldsymbol{x},t)}{\partial x_i} = D_l \frac{\partial T_l(\boldsymbol{x},t)}{\partial x_i^2} \qquad l = 1,2$$
(20)

Equation (20) is valid in each fluid region l, subject to the following matching, boundary and initial conditions:

$$T_1(\boldsymbol{\xi}, t) = T_2(\boldsymbol{\xi}, t) \quad for \quad \boldsymbol{\xi} \in S \tag{21}$$

$$D_1 \frac{\partial T_1(\boldsymbol{\xi}, t)}{\partial n_i} = D_2 \frac{\partial T_2(\boldsymbol{\xi}, t)}{\partial n_i} \quad for \quad \boldsymbol{\xi} \in S$$
(22)

$$\frac{\partial T_1(\boldsymbol{x},t)}{\partial r} \to T_0 \quad T_2(\boldsymbol{x},t) \to T_\infty \tag{23}$$

$$T_l(\boldsymbol{x},0) = f_l(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega_l, \quad l = 1,2$$
(24)

In equations (21) and (22), we enforce continuity of field variable and flux respectively, noting that the fluid velocity u_i is continuous across the interface. The temperature follows a prescribed, smoothed step initial condition given by f_l in each fluid zone (24). The smoothed step equation is given by equation (1).

The heat transfer completes the mathematical formulation of the problem. In order to evaluate the pressure field in equation (12), and the temperature field in equation (20), we require the discretisation of the domain, interface and boundaries as well as the use of a suitable numerical method, discussed in the following sections.

3 Numerical Methods

Here, we describe the solution procedure to evaluate the transient temperature field and quasi-static pressure in order to reconstruct the interface velocity with equation (19). The homogeneous pressure is solved using a boundary element method similar to that presented by [10], in which the interface is discretised with boundary elements for the solution of an integral equation; see the interface in Figure 2. Note that only 1/8th of the BEM mesh is shown here, but in practice the full interface is solved. The boundary element method described by [10] explicitly tracks the interface, providing greater accuracy compared to front capturing methods, at the expense of increased meshing computational complexity. Using a convergent (Neumann) series technique, the computational cost scales quadratically with the number of boundary elements, significantly reducing running times compared with traditional LU solvers of fully populated BEM matrices.

The temperature equation (20) and perturbed pressure equation (12) are solved via a strong form radial basis function finite collocation (RBF-FC) method that requires both internal and boundary nodes as seen in Figure 2 [11]. To decrease the solution cost for this method we use an 8 finger symmetric interface throughout the numerical simulations; this enables the RBF-FC method to make use of symmetry in the domain and solve only $1/8^{th}$ of the entire domain.



Figure 2: Thermo-viscous solution domain showing different RBF-FC operator types. The curved solid black line indicates the fluid-fluid interface, where the BEM mesh is constructed.

3.1 Homogeneous pressure - BEM

The homogeneous pressure can be represented as a sum of the pressures due to an injection potential source, Q, and the double layer potential density, $W(\mathbf{x}, \psi)$.

$$\bar{p}_1(\boldsymbol{x}) = W(\boldsymbol{x}, \psi) - \frac{1}{2\pi\beta} \ln(r), \quad \bar{p}_2(\boldsymbol{x}) = \beta W(\boldsymbol{x}, \psi) - \frac{1}{2\pi} \ln(r)$$
(25)

Here, β is the mobility ratio between the two fluids, $\beta = \overline{M}_1/\overline{M}_2$ and $W(\boldsymbol{x}, \psi)$ is the double layer potential, a function of the double layer potential density, ψ [12]. Substituting the homogeneous pressures into (13) and taking the limit at the interface of the two fluids, the following 2^{nd} kind Fredholm equation is obtained:

$$-\frac{1}{2}\psi(\boldsymbol{\xi}) + \lambda \int_{s} K(\boldsymbol{\xi}, \boldsymbol{y})\psi(\boldsymbol{y})dS_{y} = \frac{1}{Ca_{g}(1+\beta)} \left(\frac{2}{b} + \frac{\pi}{4}k(\boldsymbol{\xi})\right) + \left(\frac{\lambda}{2\pi\beta}\right)\ln(r)$$
(26)

Where, $\lambda = \frac{1-\beta}{\beta+1}$ and $K(\boldsymbol{\xi}, \boldsymbol{y})$ is the normal derivative of the fundamental solution of the laplace equation. For the solution of equation (26), we use a similar solution procedure to [10] for the isothermal Hele-Shaw cell case. We discretise the boundary into N_e cubic B-spline boundary elements using shape functions given by [13]. Along with the B-Spline representation of the interface, a 4th order Lagrangian polynomial is used to compute the local curvature, $k(\boldsymbol{\xi})$ at a nodal point. Equation (26) can be solved using a convergent series, so long as $0 \leq \beta < \infty$ [12]. Using infinitely many terms results in an analytical Neumann convergent series solution. The series can be simplified by using a finite number of terms m to truncate the solution of equation (26):

$$\psi(\boldsymbol{\xi}) = \psi_0(\boldsymbol{\xi}) + \lambda \psi_1(\boldsymbol{\xi}) + \dots + \lambda^m \psi_m(\boldsymbol{\xi})$$
(27)

The terms in equation (27) can be calculated recursively, via the following formulae:

$$\psi_0(\boldsymbol{\xi}) = -\frac{2}{Ca_g(1+\beta)} \left(\frac{2}{b} + \frac{\pi}{4}k(\boldsymbol{\xi})\right) - \left(\frac{2\lambda}{2\pi\beta}\right)\ln(r)$$
(28)

$$\psi_m(\boldsymbol{\xi}i) = 2 \int_s K(\boldsymbol{\xi}, \boldsymbol{y}) \psi_{m-1}(\boldsymbol{y}) dS_y \quad for \quad m \neq 0$$
⁽²⁹⁾

Once ψ has been calculated around the interface, the spatial derivatives of the double layer potential can be found to reconstruct the homogeneous normal velocity at the interface (and the source term for the perturbation pressure equation (12) in the corresponding zone), i.e.

$$\bar{u}_n(\boldsymbol{\xi}) = -M_1(\boldsymbol{\xi})n_i \frac{\partial \bar{p}_1(\boldsymbol{\xi})}{\partial x_i} = -M_1(\boldsymbol{\xi})n_i \left(\frac{\partial W(\boldsymbol{\xi}, \psi)}{\partial x_i} - \frac{x_i}{2\pi r^2 \beta}\right)$$
(30)

The reconstruction of the normal derivative of the double layer potential around the interface in (30) requires the evaluation of a hypersingular integral, which is handled numerically using a subtraction technique, described in more detail in [10]. As the surface interface grows, the number of boundary elements are adaptively increased to maintain a target element size.

3.2 General RBF-FC method for Multi-zone transport problems

The RBF-FC methods presented in this section cover new formulations for multi-zone steady-state and transient convection-diffusion transport problems. These methods are described in part in the works of [11][14][15].

We formulate an embedded multi-zone method for the perturbed pressure problem, where the pressure and flux matching conditions are defined explicitly in the global system. Global enforcement shows increased accuracy compared to local enforcement when considering steady-state problems. An auxiliary multi-zone method is formulated for the transient heat transfer, in which the temperature and flux matching conditions appear at the local level, and not in the global system. This aids numerical stability and the solution of high Peclet number flows present in the thermal case.

Both multi-zone methods follow very similar solution procedures, with minor differences only in the local and global system formulations, which will be highlighted. We consider a problem domain for both formulations given in Figure 2. The multi-zone convection-diffusion problems under consideration in this work can be described generally as initial boundary value problems:

$$\frac{\partial \phi_l(\boldsymbol{x}, t)}{\partial t} = L_l[\phi_l(\boldsymbol{x}, t)] + S_l(\boldsymbol{x}, t) \qquad \boldsymbol{x} \in \Omega_l$$
(31)

$$\phi_l(\boldsymbol{x}, 0) = p_l(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega_l \tag{32}$$

$$B_l[\phi_l(\boldsymbol{x},t)] = g_l(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial \Omega_l \tag{33}$$

$$C_1[\phi_1(\boldsymbol{x},t)] - C_2[\phi_2(\boldsymbol{x},t)] = h(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial\Omega_{int}$$
(34)

$$Q_1[\phi_1(\boldsymbol{x},t] - Q_2[\phi_2(\boldsymbol{x},t] = f(\boldsymbol{x}) \quad \boldsymbol{x} \in \partial\Omega_{int}$$
(35)

Here, Ω_l is the interior of zone l, $\partial\Omega_l$ is the boundary of zone l and $\partial\Omega_{int}$ is the fluid-fluid interface. Equation (32) is the initial condition for the problem. Equations (34) and (35) are the flux and solution matching conditions at the interface. For continuous flux and solution value $h(\mathbf{x}) = f(\mathbf{x}) = 0$. We take a Crank-Nicholson approximation to the time derivative to obtain modified PDE governing operators:

$$\overline{L} = I - \theta \Delta t L$$

$$\hat{L} = I + (1 - \theta) \Delta t L$$
(36)

such that

$$\overline{L}_l\left[\phi_l^{n+1}\right] = \hat{L}_l\left[\phi_l^n\right] + S_l\left(x,t\right) = \overline{S}_l(x,t) \tag{37}$$

In the equations above, L (\hat{L} and \overline{L}) and B are linear partial differential operators on the domain Ω and on the boundary $\partial \Omega$ respectively. By using this type of time stepping algorithm, the transient problem is reduced to a series of inhomogeneous steady-state problems, with the inhomogeneous term a function of the solution at the previous time step. Steady-state problems are achieved by setting \hat{L} to zero and $\overline{L} = L$ in the above equations.

The Hermitian collocation approach constructs the value of the field-variable ϕ as a weighted sum of partial differential operators applied to a set of radial basis functions that are centred on nodes ξ_j . At nodes lying on the domain boundary the boundary operator B is applied to the corresponding RBF, at nodes within the domain interior the governing PDE \overline{L} is applied, and at nodes on the interface Q and C are applied; see equation (38).

$$\phi_{l}^{n+1}(\boldsymbol{x}) = \sum_{j=1}^{NB} \alpha_{j} B_{l,\xi} \Psi\left(\|\boldsymbol{x} - \boldsymbol{\xi}_{j}\|\right) + \sum_{j=NB+1}^{NB+NI+1} \alpha_{j} \overline{L}_{l,\xi} \Psi\left(\|\boldsymbol{x} - \boldsymbol{\xi}_{j}\|\right)$$

$$+ \sum_{j=NB+NI+2}^{NB+NI+NF+2} \alpha_{j} C_{l,\xi} \Psi\left(\|\boldsymbol{x} - \boldsymbol{\xi}_{j}\|\right) + \sum_{j=NB+NI+NF+3}^{NB+NI+2NF+3} \alpha_{j} Q_{l,\xi} \Psi\left(\|\boldsymbol{x} - \boldsymbol{\xi}_{j}\|\right)$$
(38)

In equation (38), NB is the number of boundary nodes, NF is the number of flux (interface) nodes and NI is the number of internal nodes. The RBF operators in equation (38) are Hardy Multiquadric RBF interpolants. We use a global, non-dimensional version of the shape parameter c^* , which is related to the local shape parameter c through $c^* = c/\Delta$, where Δ is the average nodal separation in the local stencil.

In the RBF-FC method, the global domain is broken down into a series of overlapping local stencils, formed at every interior node in the domain. Enforcing the PDE system (31 - 35) at a set of test locations equal to the set of functional centres in a local stencil in zone l, a collocation matrix can be formed at each of the N interior nodes. For an interior system in zone l, with strictly interior nodes that do not fall on interface this takes the form:

$$\begin{bmatrix} B_{l,x}B_{l,\xi}\left[\Psi_{ij}\right] & B_{l,x}\overline{L}_{l,\xi}\left[\Psi_{ij}\right] \\ \overline{L}_{l,x}B_{l,\xi}\left[\Psi_{ij}\right] & \overline{L}_{l,x}\overline{L}_{l,\xi}\left[\Psi_{ij}\right] \end{bmatrix} \alpha_{j} = \begin{bmatrix} g_{l,i} \\ \overline{S}_{l,i} \end{bmatrix}$$
(39)

Where $B_{l,x}$ is the application of operator B in zone l at the test location x. Through the reconstruction of $\phi_l(\boldsymbol{x})$ at the centrepoint of each local system using (38), a series of N simultaneous equations are formed for the N unknown values. Values of $\phi_l(\boldsymbol{x})$ are then found through the solution of this sparse matrix system. The resulting values of $\phi_l(\boldsymbol{x})$ can then be fed back into the local data vectors (RHS of (39)), allowing the value of $\hat{L}_l[\phi^n]$ to be reconstructed at the end of each timestep and the transient problem iterated. Any other spatial/temporal derivatives can also be reconstructed at the end of each timestep.

3.3 Steady state pressure - Embedded RBF-FC multi-zone method

Figure 2 shows the boundary conditions and an example nodal discretisation for the solution of the perturbed pressure in equation (12). As the present Darcy formulation is quasi-static, the perturbed pressure equation (12) can be described as a steady-state multi-zone transport boundary value problem of the form:

$$L_{l,x}[\tilde{p}_l(\boldsymbol{x})] = -S_l(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega_l \tag{40}$$

$$B_{l,x}[\tilde{p}_l(\boldsymbol{x})] = g_l(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial \Omega_l \tag{41}$$

$$C_{1,x}[\tilde{p}_1(\boldsymbol{x})] - C_{2,x}[\tilde{p}_2(\boldsymbol{x})] = h(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial\Omega_{int}$$

$$\tag{42}$$

$$Q_{1,x}[\tilde{p}_1(\boldsymbol{x})] - Q_{2,x}[\tilde{p}_2(\boldsymbol{x})] = f(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial\Omega_{int}$$

$$\tag{43}$$

Where:

$$L_{l,x} = M_l(\boldsymbol{x}) \frac{\partial^2}{\partial x_i^2} + \frac{\partial \tilde{M}_l(\boldsymbol{x})}{\partial x_i} \frac{\partial}{\partial x_i}, \quad S_l(\boldsymbol{x}) = \frac{\partial \tilde{M}_l(\boldsymbol{x})}{\partial x_i} \frac{\partial \bar{p}_l(\boldsymbol{x})}{\partial x_i} \qquad \boldsymbol{x} \in \Omega_l$$
(44)

$$B_{l,x}^{DU} = B_{l,x}^{DK} = 1, \ B_{l,x}^{N} = n_i \frac{\partial}{\partial x_i} \quad g_l^{DU}(\boldsymbol{x}) = \tilde{p}_l(\boldsymbol{x}), \ g_l^{DK}(\boldsymbol{x}) = g_l^{N}(\boldsymbol{x}) = 0 \qquad \boldsymbol{x} \in \partial \Omega_l$$
(45)

$$C_{l,x} = 1, \ Q_{l,x} = M_l(\boldsymbol{x}) n_i \frac{\partial}{\partial x_i}, \ h_x = 0, \ f_x = \left(\beta \tilde{M}_2(\boldsymbol{\xi}) - \tilde{M}_1(\boldsymbol{\xi})\right) \frac{\partial \bar{p}_1(\boldsymbol{\xi})}{\partial n_i} \qquad \boldsymbol{x} \in \partial \Omega_{int}$$
(46)

In equations (40) to (46), subscript l refers to the zone. L_l , B_l and C_l/Q_l are linear partial differential operators on the domain Ω_l , the boundary $\partial \Omega_l$ and the fluid-fluid interface $\partial \Omega_{int}$ respectively. $S_l(\boldsymbol{x})$ is the source term coming from the homogeneous pressure $\bar{p}(\boldsymbol{x})$ in domain Ω_l . Superscript DU, DK and N refer to Dirichlet unknown, Dirichlet known and Neumann boundary operators respectively.

We utilise the embedded multi-zone RBF-FC method to discretise the system of equations (40) to (43). For the embedded method, two stencils are formed at each interface point, with the interface flux and field variable appearing as an unknown in the corresponding data vector. At the stencil centrepoint the corresponding zone PDE value \overline{L}_l is enforced. An example collocation system for zone 1 is given below:

$$\begin{bmatrix} B_{1,x}B_{1,\xi} & B_{1,x}\overline{L}_{1,\xi} & B_{1,x}C_{1,\xi} & B_{1,x}Q_{1,\xi} \\ \overline{L}_{1,x}B_{1,\xi} & \overline{L}_{1,x}\overline{L}_{1,\xi} & \overline{L}_{1,x}C_{1,\xi} & \overline{L}_{1,x}Q_{1,\xi} \\ C_{1,x}B_{1,\xi} & C_{1,x}\overline{L}_{1,\xi} & C_{1,x}C_{1,\xi} & C_{1,x}Q_{1,\xi} \\ Q_{1,x}B_{1,\xi} & Q_{1,x}\overline{L}_{1,\xi} & Q_{1,x}C_{1,\xi} & Q_{1,x}Q_{1,\xi} \end{bmatrix} \alpha_{j} = \begin{bmatrix} g_{1,i} \\ \overline{S}_{1,i} \\ C_{1}[\phi(\boldsymbol{x},t] \\ Q_{1}[\phi(\boldsymbol{x},t] \end{bmatrix}$$
(47)

The data vectors contain the unknown interior solution values $g_{1,i}$ and the unknown interface solution and flux values $C_1[\phi(\boldsymbol{x}, t]]$ and $Q_1[\phi(\boldsymbol{x}, t]]$ respectively. From these the corresponding zone 2 interface solution and flux values can be calculated.

After the solution of the resulting sparse global system, the normal perturbation velocity $\tilde{u}_n(\boldsymbol{\xi})$, at a location $\boldsymbol{\xi}$ on the interface can be reconstructed using the nearest overlapping system, i.e.

$$\tilde{u}_n(\boldsymbol{\xi}) = -M_1(\boldsymbol{\xi}) \frac{\partial \tilde{p}_1(\boldsymbol{\xi})}{\partial n_i}$$
(48)

The interface is then displaced via a forward Euler time stepping scheme, using a timestep size of Δt .

3.4 Transient heat transfer - Auxiliary RBF-FC multi-zone method

To solve the heat transfer in equation (20) and compute the viscosity field (5) for use in the pressure field calculations, we use an Auxiliary RBF method with domain shown in Figure 2. The heat equation can be described by a transient multi-zone transport boundary value problem of the form:

$$\overline{L}_{l,x}^{n+1}[T_l^{n+1}(\boldsymbol{x})] = \overline{S}_{l,x}^n \qquad \boldsymbol{x} \in \Omega_l$$
(49)

$$B_{l,x}^{n+1}[T_l^{n+1}(\boldsymbol{x})] = g_l(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial \Omega^l$$
(50)

$$C_{1,x}^{n+1}[T_1^{n+1}(\boldsymbol{x})] - C_{2,x}^{n+1}[T_2^{n+1}(\boldsymbol{x})] = h(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial\Omega_{int}$$
(51)

$$Q_{1,x}^{n+1}[T_1^{n+1}(\boldsymbol{x})] - Q_{2,x}^{n+1}[T_2^{n+1}(\boldsymbol{x})] = f(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial\Omega_{int}$$
(52)

Where:

$$\overline{L}_{l,x}^{n+1} = I - \theta \Delta t \left(D_{l,x}^{n+1}(\boldsymbol{x}) \frac{\partial^2}{\partial x_i^2} + u_{l,x}^n \frac{\partial}{\partial x_i} \right) \qquad \boldsymbol{x} \in \Omega_l$$
(53)

$$\overline{S}_{l,x}^{n} = I + (1-\theta) \,\Delta t \left(D_{l,x}^{n}(\boldsymbol{x}) \frac{\partial^{2} T_{l}^{n}(\boldsymbol{x})}{\partial x_{i}^{2}} + u_{l,x}^{n} \frac{\partial T_{l}^{n}(\boldsymbol{x})}{\partial x_{i}} \right) \qquad \boldsymbol{x} \in \Omega_{l}$$
(54)

$$B_{l,x}^{DU,n+1} = B_{l,x}^{DK,n+1} = 1, \ B_{l,x}^{N,n+1} = n_i \frac{\partial}{\partial x_i} \qquad \boldsymbol{x} \in \partial \Omega_l \tag{55}$$

$$g_{l,x}^{DU,n+1}(\boldsymbol{x}) = T_l^{n+1}(\boldsymbol{x}), \ g_{1,x}^{DK,n+1}(\boldsymbol{x}) = T_0, \ g_{2,x}^{DK,n+1}(\boldsymbol{x}) = T_{\infty}, \ g_{l,x}^{N,n+1}(\boldsymbol{x}) = 0 \qquad \boldsymbol{x} \in \partial\Omega_l$$
(56)

$$C_{l,x}^{n+1} = 1, \quad Q_{l,x}^{n+1} = D_l(\boldsymbol{x}) n_i \frac{\partial}{\partial x_i}, \quad h_x = 0, \quad f_x = 0 \qquad \boldsymbol{x} \in \partial \Omega_{int}$$
(57)

In equations (49) to (57), subscript l refers to the zone and superscript n and n + 1 refer to the timestep in question. L_l , B_l and C_l/Q_l are linear partial differential operators on the domain Ω_l , the boundary $\partial \Omega_l$ and the fluid-fluid interface $\partial \Omega_{int}$ respectively. Superscript DU, DK and N refer to Dirichlet unknown, Dirichlet known and Neumann boundary operators respectively.

We utilise the Auxiliary multi-zone RBF-FC method to discretise the system of equations (49) to (52). In contrast to the Embedded method, the auxiliary multi-zone method collocates only one stencil at each interface point, enforcing operators from zone 1 and zone 2 in the same matrix system, i.e.

$$\begin{bmatrix} B_{1,x}B_{1,\xi} & 0 & B_{1,x}\overline{L}_{1,\xi} & 0 & B_{1,x}C_{1,\xi} & B_{1,x}Q_{1,\xi} \\ 0 & B_{2,x}B_{2,\xi} & 0 & B_{2,x}\overline{L}_{2,\xi} & B_{2,x}C_{2,\xi} & B_{2,x}Q_{2,\xi} \\ \overline{L}_{1,x}B_{1,\xi} & 0 & \overline{L}_{1,x}\overline{L}_{1,\xi} & 0 & \overline{L}_{1,x}C_{1,\xi} & \overline{L}_{1,x}Q_{1,\xi} \\ 0 & \overline{L}_{2,x}B_{2,\xi} & 0 & \overline{L}_{2,x}\overline{L}_{2,\xi} & \overline{L}_{2,x}C_{2,\xi} & \overline{L}_{2,x}Q_{2,\xi} \\ C_{1,x}B_{1,\xi} & -C_{2,x}B_{2,\xi} & C_{1,x}\overline{L}_{1,\xi} & -C_{2,x}\overline{L}_{2,\xi} & C_{1,x}C_{1,\xi} - C_{2,x}C_{2,\xi} & C_{1,x}Q_{1,\xi} - C_{2,x}Q_{2,\xi} \\ Q_{1,x}B_{1,\xi} & -Q_{2,x}B_{2,\xi} & Q_{1,x}\overline{L}_{1,\xi} & -Q_{2,x}\overline{L}_{2,\xi} & Q_{1,x}C_{1,\xi} - Q_{2,x}C_{2,\xi} & Q_{1,x}Q_{1,\xi} - Q_{2,x}Q_{2,\xi} \end{bmatrix} \alpha_{j} = d$$

$$(58)$$

Where,

$$d = \begin{bmatrix} g_{1,i} & g_{2,i} & \overline{S}_{1,i} & \overline{S}_{2,i} & h_i & f_i \end{bmatrix}'$$
(59)

In (58) and (59), all operators are at evaluated at timestep n + 1. In the auxiliary system, the unknown solution flux's do not appear in the data vector, with f_i instead enforcing the flux matching condition locally. Similarly, the field variable matching condition appears locally in the form h_i . The PDE centres present at the system centrepoints of the two embedded systems are not apparent in the auxiliary system. This is because the local enforcement of the matching conditions provides a strong collocation at the centrepoint, which already takes into account the multi-zone transport scalars that appear in the PDE centres.

After the formation of collocation systems and reconstruction of the temperature at the system system centrepoints, the solution of the resulting sparse global system can be obtained for timestep n + 1. Using the newly computed temperature field, the viscosity variation at timestep n + 1 using equation (5) can be computed. After the pressure and velocity field have been calculated with equation (12) and the interface moved to its new position, we can calculate the values of $S_l^n(\boldsymbol{x})$ in the new domain using the updated velocity/diffusivity and the old local systems.

A common issue encountered in fixed mesh moving boundary problems is that of ghost nodes [16][17]. This occurs when the moving interface moves over an interior node, and the zone switches from 1 to 2 between timestep n and n + 1. This means the value of $S_l^n(\boldsymbol{x})$ needs to be calculated for the node, but at timestep n the node was positioned in zone 2, meaning there is no data to reconstruct $S_l^n(\boldsymbol{x})$ here. Transport data such as the fluid velocity and diffusivity are only known in the specified regions at a specified timestep, and do not exist outside of these regions, meaning the zone 1 fluid velocity at timestep n cannot be reconstructed accurately by the numerical method in zone 2.

Generally, some form of extrapolation is used to reconstruct the data needed from the previous timestep in a small band outside of the interior zone [18]. However, this limits the convergence and accuracy of the numerical method to that of the extrapolation scheme, which can be very poor when the data varies sharply over the interface.

In the auxiliary method presented here, we alleviate the issue of data extrapolation by simply translating nodes so that all nodes apart from the interface nodes will exist in a region where they also existed at timestep n. As interface nodes only require reconstruction of the solution or flux, which do not depend on previous timestep data, no extrapolation is needed for these even though they switch between zones at each timestep.

3.5 Quadtree dataset generation

All of the RBF-FC methods described above require a distribution of nodal points throughout the domain. In this work, we use an adaptive method for choosing the position of nodal points in the domain based on a quadtree data structure, with an example shown in Figure 2. The adaptive quadtree data structure has been previously implemented with the RBF-FC method for single-zone transport problems by [14]. In the present work, we use a similar quadtree dataset that is based on the 1-1 stencil configuration (see Fig. 6 & 7 in [14]). Nodal points close to the interface are deformed to lie directly on the interface so that flux and field variable matching conditions can be applied consistently.

The quadtree is initialised using a distance based scheme to cluster cells around the boundary element interface, in which a cell will split if the following equality is met: $D_cB_t > R_c$. Here, D_c refers to the diagonal length of the cell, R_c is the distance from the cell centre to the closest point on the interface and B_t is the band thickness. After initialisation, the cells closest to the interface are deformed such that their vertices lie directly on the interface and the centres lie at the geometric centre of the cell. Vertices move in a purely orthogonal direction to maintain spatial consistency.

After the quadtree has been generated, the nodal points for the RBF-FC local systems can be created based on the vertices and cell centres of the leaf cells. A simple tree search can be performed, with nodal locations picked out as the tree is recursively traversed, and grouped into local stencil configurations. The quadtree dataset dynamically adapts to the moving interface as it displaces with time, with nodes being added, removed and 'snapped' in accordance to the new interface position at each timestep.

3.6 Coupled pressure/flow - Heat Transfer Algorithm

We solve the coupled thermo-viscous problem using the numerical methods described in the previous section following the algorithm below:

- 1. Set-up the initial interface using 8 finger symmetric perturbation, $r = 1 + \epsilon_0 \cos(8\theta)$. Set-up the initial temperature field using a smoothed step function over the interface using equation (1), $\delta = 1$. Generate the boundary element mesh and embedded RBF-FC nodal arrangement.
- 2. Solve the initial pressure/velocity field using the initial temperature field and interface, through equations (30), (48) and (19).
- 3. Displace the interface using a forward Euler timestep, i.e $\Delta x_i = n_i U_n \Delta t$. Advance $t = t + \Delta t$.
- 4. Generate and update auxiliary/embedded RBF-FC nodes using quadtree dateset for new interface position. Snap relevant nodes onto the interface.
- 5. Solve the transient heat transfer equation (20) using the auxiliary RBF-FC method.
- 6. Use the temperature field at the new interface location to calculate the viscosity using (5).
- 7. Solve the pressure/velocity field using the new viscosity field and interface, through equations (30), (48) and (19).
- 8. Repeat steps 3 7 until the end simulation time is reached.

4 Numerical Validation

In order to validate the coupled numerical algorithm, we solve the displacement of an initially circular interface, with zero perturbation. Perturbations will only grow after a long period of time due to numerical error, meaning that the circular interface should propagate with an analytical radius $R_a(t)$ that grows with time according to:

$$R_a(t) = \sqrt{\frac{t+\pi}{\pi}} \tag{60}$$

The evolution in (60) is purely due to the source injection at the origin, with a radial velocity $u_s = Q/(2\pi R_a)$. The comparison of the radial evolution given by the numerical scheme and the analytical value above will highlight the accuracy of the proposed scheme. Although the homogeneous velocity will be that coming purely from the source term (as the boundary element method will give a net zero velocity as there is no perturbation), the perturbed velocity will still contribute and come from the embedded RBF-FC method with viscosity field from the transient heat transfer. Therefore, since the BEM has already been previously validated in [10], the circular displacement example here will provide a good validation of the complete coupled scheme.

Case	$T_0(^{\circ}C)$	$T_{\infty}(^{\circ}C)$	$\overline{b}(m)$	$\overline{\mu_1}(Pa.s)$	$\overline{\mu_2}(Pa.s)$	$\overline{M_1}(m^3.s/kg)$	$\overline{M_2}(m^3.s/kg)$
1	50	35	$5.38 \ge 10^{-5}$	$6.89 \ge 10^{-5}$	$5.78 \ge 10^{-4}$	$3.49 \ge 10^{-6}$	$4.17 \ge 10^{-7}$
2	35	50	$6.14 \ge 10^{-5}$	$8.41 \ge 10^{-5}$	$7.54 \ge 10^{-4}$	$3.74 \ge 10^{-6}$	$4.17 \ge 10^{-7}$
3	70	35	$4.64 \ge 10^{-5}$	$5.31 \ge 10^{-5}$	$4.31 \ge 10^{-4}$	$3.38 \ge 10^{-6}$	$4.17 \ge 10^{-7}$
4	35	70	$6.14 \ge 10^{-5}$	$8.41 \ge 10^{-5}$	$7.54 \ge 10^{-4}$	$3.74 \ge 10^{-6}$	$4.17 \ge 10^{-7}$
5	35	35	$6.14 \ge 10^{-5}$	$8.41 \ge 10^{-5}$	$7.54 \ge 10^{-4}$	$3.74 \ge 10^{-6}$	$4.17 \ge 10^{-7}$

Table 1: Summary of fluid properties and injection/resident temperatures.

Throughout this section and the results section we use 5 different cases representing different injection scenarios using supercritical CO_2 and brine, summarised in Table 1. We change the plate spacing b between cases, so that the outer fluid mobility is equal for all cases. This means that the same injection flux will create the same global capillary number regime and effects arising from the spatially varying viscosity are easier to distinguish. This is discussed in more detail in the isothermal results section.

For cases 1 and 2 the rates of change of viscosity with non-dimensional temperature are $\alpha_1 = 0.198564060$, $\alpha_2 = 0.26266280$. For cases 3 and 4 $\alpha_1 = 0.458335046$, $\alpha_2 = 0.539014472$. This difference is due to the non-dimensional temperatures in each case, since the range $|T_0 - T_\infty|$ changes, affecting equation (5). The dimensional viscosity - temperature profiles are however the same for each case. We use a supercritical CO_2 and Brine thermal diffusivity of $D_1 = 0.0001$ and $D_2 = 0.0003$ respectively.

Given the symmetry of the problem, we use a triangular solution domain for the RBF-FC methods, similar to that shown in Figure 2. The mesh/timestep parameters that follow are used for all results, with all solutions being both mesh and timestep independent. For the perturbed pressure solution, the embedded RBF-FC method has inner and outer boundaries at $r_0 = 0$ and $r_1 = 32$ respectively. The minimum quadtree cell level is 3 with a maximum level of 10, meaning $\Delta x_{min} = 0.03125$. $B_t = 3$, $c^* = 90$. For the auxiliary RBF-FC method for heat transfer, the inner boundary is slightly displaced from the origin, so that the infinite fluid velocity at the origin does not have to be included. Therefore, $r_0 = 0.03125$ and $r_1 = 8$, with a minimum quadtree cell level of 4 and a maximum level of 8, $\Delta x_{min} = 0.03125$. $B_t = 6$, $c^* = 100$. In the boundary element method the full interface is solved, around which a target element size is maintained throughout the simulations at $\Delta x \approx 0.06$. This ensures that the RBF-FC nodal refinement is around twice that of the BEM around the interface. The timestep size for the coupled problem is $\Delta t = 0.02$.

To quantify the difference between the numerical and analytical circular displacements, we use the L_1 and L_2 errors, given by:

$$L_1 \text{ error} = \frac{\sum_{1}^{N_e} |R_n - R_a|}{N_e R_a} \quad L_2 \text{ error} = \sqrt{\frac{\sum_{1}^{N_e} (R_n - R_a)^2}{N_e R_a^2}}$$
(61)

Here, R_n is the numerical radial distance from a point on the interface to the source injection, N_e is the number of boundary element nodes where the analytical/numerical radii are compared. The validation results for the 4 non-isothermal cases are presented in Table 2. All cases show a small relative error from the analytical values at t = 10, with the largest error coming from case 3. This is as expected as this case is the most unstable due to the large temperature variation through the domain. However, the L_2 error of 1.59×10^{-2} at t = 10 (after 500 timesteps) is still small, with the main error coming from the transient temperature field which has a very high thermal Peclet number, $Pe = 1/D_1 = 1 \times 10^4$.

Case	$T_0(^{\circ}C)$	$T_{\infty}(^{\circ}C)$	L_1 Error	L_2 Error
1	50	35	$8.45 \ge 10^{-4}$	$6.85 \ge 10^{-4}$
2	35	50	$1.30 \ge 10^{-3}$	$1.02 \ge 10^{-3}$
3	70	35	$1.87 \ge 10^{-2}$	$1.59 \ge 10^{-2}$
4	35	70	$3.59 \ge 10^{-3}$	$2.57 \ge 10^{-3}$

Table 2: Solution errors between analytical and numerical circle positions at t = 10, $Ca_g = 2000$.

Figure 3 shows the interfacial displacement and temperature field for case 1 at t = 10. This highlights the high Peclet number, as the temperate field shows very little diffusion and the interface lies at a temperature close to the initial temperature condition, i.e. the initial condition has been almost purely advected. The analytical and numerical solutions are indistinguishable, showing the accuracy of the proposed scheme and the validation of the coupled thermo-viscous problem.



Figure 3: Interface plot at t=10 for case 1. Solid line - analytical solution, circles - numerical solution. The interface overlays a contour plot of the temperature field.

4.1 Isothermal displacement

In this section we demonstrate the effect of changing both the injection and resident fluid temperatures in an isothermal regime with homogeneous viscosity (and mobility). Using three different isothermal temperatures regimes of $35^{\circ}C$, $50^{\circ}C$ and $70^{\circ}C$, we present interface plots at t = 35 for cases with free and fixed mobilities in Figure 4. The forced mobility cases have b changed between the $35^{\circ}C$, $50^{\circ}C$ and $70^{\circ}C$ regimes so that M_2 is forced to be the same value. In the free mobility cases, b is kept the same between each temperature regime, meaning the homogeneous values of M_2 are different for the $35^{\circ}C$, $50^{\circ}C$ and $70^{\circ}C$ cases, since the viscosity is temperature dependent.



Figure 4: Interface plots at t=35 for isothermal displacements. — $70^{\circ}C,$ – – – $50^{\circ}C$, – – – $35^{\circ}C$

Figure 4 highlights the effects of changing the global capillary number. In the cases with a fixed mobility, the global capillary number is $Ca_g = 4000$, resulting in an almost identical interfacial displacement for the three different temperature regimes. The only major difference is in the finger bases, with the bases in the $35^{\circ}C$ case being closest to the origin. This is because the mobility ratio is slightly higher at a temperature of $35^{\circ}C$ compared to the $50^{\circ}C$ and $70^{\circ}C$ cases, meaning the finger bases stagnate more readily.

In the free mobility cases, M_2 changes between each temperature regime. This means the corresponding global capillary numbers are $Ca_g = 7006$, $Ca_g = 5368$ and $Ca_g = 4000$ for the $35^{\circ}C$, $50^{\circ}C$ and $70^{\circ}C$ cases respectively. The fluid viscosities at $35^{\circ}C$ are higher in comparison to the other cases, meaning with a fixed injection flux Q, the capillary number is correspondingly higher. This creates a much more unstable regime, as the surface tension is less able to resist the viscous movement of the fluids. Viscous forces are larger in the $35^{\circ}C$ case, meaning the finger fronts bifurcate into more fingers than in the higher temperature cases (four fingers compared to three/two).

In the following non-isothermal section, we change the plate separation between the 5 different temperature cases (as seen in table 1), so that the homogeneous mobility is equal in each case. As the injection/resident temperature changes between each case, the homogeneous mobility and global capillary number would also change if we had a fixed plate separation, resulting in a different flow regime. This could be wrongly misinterpreted as an effect due to the inhomogeneous viscosity, as opposed to globally changing the capillary number. We have demonstrated the effects of a global change in capillary number, and now look to provide insight into the cases of spatially varying viscosity and local changes in capillary number caused by this inhomogeneous viscosity variation.

4.2 Non-isothermal displacement

We now consider the cases of non-isothermal displacement given in table 1. All cases are run with the mesh/timestep parameters discussed in the previous section. In Figure 5, the interfacial displacements at t = 30 can be seen for the different temperature regimes at capillary numbers of $Ca_g = 2000$ and $Ca_g = 4000$. In both capillary number regimes, negative temperature gradients in the domain have accelerated the interface, promoting bifurcation and leading to a more ramified structure seen by the dashed and solid red lines in Figure 5. This acceleration is due to the perturbed pressure, which is strong and positive in the case of a negative temperature gradient. The source term, effective velocity and diffusivity term in equation (12) are positive, meaning that the perturbed pressure is generated strongly near the interface and

convected/diffused radially, decreasing in value. This creates a positive velocity contribution which helps to accelerate the interface.



Figure 5: Interface plots at t = 30 for varying temperature displacements. - - Case 1, - - Case 2, - Case 3, - Case 4, - Case 5.

The opposite effect can be seen by the positive temperature gradient cases, where a cold fluid is injected into a hotter resident fluid. In the dashed and solid blue line cases in Figure 5, the interfaces are more stable and less ramified than the isothermal injection case. At a capillary number of $Ca_g = 2000$, cases 2 and 4 still have not bifurcated, with the displacement occurring in a much more stable regime. In case 2 in the $Ca_g = 4000$ regime, the interface shows a very similar displacement to the isothermal case, and appears not to have shown as much stabilisation as the other positive temperature gradient cases. At this very high capillary number, bifurcation occurs readily when the finger front flattens, and the positive temperature gradient here has not been enough to completely stabilise the interface. When the resident fluid temperature is increased in case 4, the bifurcation has been delayed and the interface appears more stable. However, the interface will split into 3 fingers late in time, meaning the bifurcation regime will change from the two finger split.

The finger bases in Figure 5 show corresponding behaviour due to the acceleration or deceleration of the finger fronts. In cases where acceleration has occurred, the finger bases lie closer to the origin, whereas in cases of deceleration, they are pulled closer to the finger fronts due to the stabilisation effects. The perturbed pressure contribution causes the relative difference between the base positions. The temperature at the bases is higher for the hot-cold injection cases, meaning the mobility is also higher. However, if this was the primary effect, then the bases would be further displaced into the domain, which is not the case in Figure 5. The higher temperatures do have a lower mobility ratio between the fluids, however this is not enough to explain the large difference in the base position, when a similar difference in mobility ratios caused only a very minor change in base position in Figure 4. The primary mechanism can therefore be attributed to the movement of the finger front, which when accelerated by the negative temperature gradients causes a preferential movement of the flow to the finger fronts, which aids in the stagnation of the finger bases. When the fronts are decelerated and stabilised, the flow can move more significantly towards the finger bases and helps to push them forwards into the domain. In Figure 5b, there is no case 3 interface, as the high temperature regime and high capillary number create a very unstable regime that creates too large an error from the heat transfer in the auxiliary RBF-FC method. The Peclet number here is 2×10^4 , exhibiting an almost shock like temperature profile that is extremely hard to capture. This coupled with the significant acceleration of the interface causes a highly unstable bifurcation where numerical error is quite large and the resulting number of fingers cannot accurately be predicted. In the lower temperature case 1 in Figure 5b, the interface has been accelerated significantly and shows much greater difference compared to the isothermal



Figure 6: Interfaces at t = 30 overlaying contour plots of the temperature field.

case, than in the corresponding lower capillary number regime. The pinching in of the finger sides is typical of a side branching regime, where four fingers will likely form when the interface progresses further in time.

Even for the most heavily stabilising cases in Figure 5, bifurcation is still expected once the temperature field diffuses out with time. If the viscosity contrast is initially unfavourable, a stabilising temperature field will never be able to completely suppress the fingering regime with time, as the field diffuses. This was also shown to be true in miscible cases by [4], in which if the convective species causes a destabilising flow regime, the diffusing (or more diffusive) regime would never be able to completely thwart the destabilisation given enough time.

Figure 6 shows contour plots of the temperature fields for the corresponding cases in Figure 5. All cases exhibit highly convective temperature profiles, with diffusion only dominating in places where localised stagnation has occurred. At the finger bases and the bifurcated finger fronts, the interface is displacing very slowly, to the point where it almost stagnates. The high temperature here quickly diffuses due to the large temperature gradients, raising the temperature to near that of the injection temperature. At areas where the interface is still moving with significantly velocity, the interface temperature sits closer to that of the resident fluid.

To highlight the temperature evolution with time, the temperature profile at sections through the finger front and finger base for case 1 at $Ca_g = 2000$ have been plotted in Figure 7. The initial temperature profile exhibits a smoothed step profile over the interface (see equation (1)), which lies at a temperature exactly halfway between the injection and resident, seen in the first profiles in Figure 7. After this, the temperature initially falls in Figure 7b, as the finger front is in a highly convective regime, with the temperature field diffusing over the finger front. The temperature somewhat lags the fluid front, as it is convected with the same fluid velocity but diffuses through the finger sides and front where the temperature gradient is high, allowing the temperature to fall. The second profile in Figure 7b shows a kink at the interface (red circle).



Figure 7: Temperature plot sections taken parallel to the finger front (along y = 0) and base (along $y = x \cdot tan(\pi/8)$). Taken at time intervals $\Delta t = 5$ between $0 \le t \le 30$. Circles correspond to the interface locations. $Ca_q = 2000$.

This is due to the multi-zone nature of the flow, which in the early timesteps causes the interface to sharply vary so that the flux matching condition is met. As the diffusivity increases by a factor of three across the interface, the temperature gradient must decrease by factor of three, so that the heat fluxes match. This is why the kink occurs and decreases the temperature gradient to the right of the interface.

The kink seen in the second profile in Figure 7b quickly smooths as the interface progresses and the temperate diffuses. After t > 15 in Figure 7b, the front starts to bifurcate and the temperature increases rapidly. The front velocity decreases, (the profiles get closer together for the same Δt) and diffusion becomes more significant, raising the temperature above that of the initial condition. At the finger bases in Figure 7a, the temperature rises above that of the initial continually from the start of the simulation. The base displaces with a much lower velocity than the front, meaning diffusion has more effect. Very early into the simulation the finger base has almost stagnated and reaches a steady-state temperature. This is slightly below that of the injection temperature, as the outer fluid diffusivity is higher than the inner diffusivity, meaning heat will be diffused from the interface more strongly in the outer zone, which drops the temperature.

The multi-zone temperature profile is highlighted further in Figure 8a. Here, the stabilising case 2 has a temperature profile over the interface that is still in the highly convective regime. The flux matching condition can be clearly seen in the blue line, where the outer temperature gradient is smaller than the inner temperature gradient at the interface. In case 1, the rapid drop in temperature appears to happen before the interface position (red circle). This is because the finger has bifurcated, and is now in a diffusive regime. The rapid drop in temperature at the point just before the interface is due to the temperature 'stagnating' here as it is diffused into the outer region. The temperature profile has been interpolated between solution points, and although it appears as only C^0 continuous, it is continuous with a sharp gradient. The coarse interpolation makes it appear discontinuous as the solution changes so rapidly. At the interface point however, the solution is only C^0 continuous, as the normal derivative of the temperature field is discontinuous here.

The viscosity field in Figure 8b highlights the local effects of the spatially varying viscosity. Here, in the



Figure 8: (a) Temperature plots taken parallel to the finger front (along y = 0) at t = 30. (b) Corresponding fluid viscosity. — Case 1, — Case 2. $Ca_g = 2000$.

hot-cold injection (case 1), the viscosity contrast is decreased at the interface, whereas in case 2, the viscosity contrast is increased. A larger viscosity contrast across the interface in an isothermal case makes the interface more unstable, as can be seen by high mobility ratio cases in previous works [10][19]. However, in the non-isothermal cases here, the localised changes in viscosity contrast do not seem to affect the displacement significantly. The larger viscosity contrast over the interface in case 2 has not caused the interface to be more unstable, as globally the interface has been stabilised. The macro-scale spatial variance in viscosity has caused the interface to be accelerated or decelerated, not the local change in viscosity contrast that can be seen over the interface. If the interface is already unstable (i.e. there is an increase in viscosity over the interface), then modifying this viscosity jump by a small amount will not affect the displacement as much as the spatially varying viscosity does in a global sense.

The viscosity only appears in the pressure jump at the interface due to the non-dimensional scaling of the problem, and appears in homogeneous form. Therefore the pressure jump condition will not be affected by a change in the inhomogeneous viscosity at the interface, meaning the homogeneous velocity is only affected by the small change in mobility at the interface. The large difference comes from the perturbed velocity, which does not depend significantly on the precise local contrast of viscosity over the interface.



Figure 9: Interface temperature scatter plots at t = 30. $Ca_q = 4000$.

The general effects of acceleration/deceleration and corresponding promotion/hindering of bifurcation are due to the temperature (and hence mobility) variation around the interface, highlighted in Figure 9. The temperature variation in case 1 from the finger base to the front is negative and has a larger absolute variation around the interface than that in case 2. This causes the positive perturbed velocity and the acceleration of the interface beyond the isothermal case. Conversely, the positive temperature gradient in case 2 aids the stabilisation of the interface, with it appearing much less ramified than case 1. This macroscopic temperature variation causes the overall difference in displacement regimes, with localised changes in temperature (as seen on the base in case 1 for example) having a very small effect. Besides, if the local effects dominated, then the Hele-Shaw approximation would break down as variations in zonal mobility have to occur on a length scale much larger than that of the plate separation.

4.2.1 Thermal diffusion effects

In this section, we explore the role of thermal diffusion in greater detail. In previous sections, the role of diffusion has been limited as the flow has been highly convective. Thermal diffusion only significantly affected the temperature at the finger bases and fronts, once they had almost stagnated, by which point the temperature effects are minimal as the interface velocity is very small.

In Figure 10 we have increased the diffusivity in both zones by a factor of 10 and 100 compared to those in Figure 5. Figures 10a and 10b have the same ratio of thermal diffusivity as before, but in Figures 10c and 10d, the diffusivity ratio has been inverted, with the inner fluid now having the higher diffusivity. In all cases, the increased diffusivity weakens the effects of a spatially varying viscosity. In the hot-cold injection cases (left hand plots), the increased diffusivity causes less acceleration of the interface and the destabilisation is less pronounced. When the diffusivity is 100x that of the base case, the interfaces in the left-hand plots tend to a similar three finger bifurcation. This is surprising, as the isothermal case has a two finger bifurcation, and it should tend towards this. However, even at 100x diffusivity the Peclet number is Pe = 100, meaning the flow is still largely convective.

In the reversed diffusivity case in Figure 10c, the interfaces all show similar three finger bifurcations, whereas in the plot above with the physical fluid diffusivities, there is a transition to three fingers as the diffusivity is increased. This is because in the reversed diffusivity case the outer fluid diffusivity is lower, meaning the temperature at the front is increased. This decreases the viscosity and lowers the local capillary, reducing the destabilising effects of the viscous flow. In the physical diffusivity case, the temperature is lowered by the high thermal diffusivity in the outer region, meaning the viscosity is higher and the front is more unstable.

In the cold-hot injection cases (right hand side plots in Figure 10), the increased diffusion has led to a slightly more unstable displacement regime. The stabilising regime caused by positive temperature gradients in the domain has been weakened. However, in this generally stable regime, increasing the diffusivity has had much less effect in comparison to the destabilising cases in the right of Figure 10.

The increased diffusivity can be seen in the temperature contour plots in Figure 11. The temperature field in the 100x diffusivity case bears very little resemblance to the three finger interface pattern, instead resembling the initial condition that has been uniformly convected and diffused. However, even though the temperature field is so diffuse, the macroscopic distribution of temperature around the interface has not been significantly affected in comparison to the lower diffusivity case, highlighted in Figure 12. The temperature variations show remarkably similar patterns, even though the temperature field is much more diffuse.

Although the temperature field in Figure 11b looks very different to that in Figure 11a, the Peclet number is still over unity, meaning the displacement will still be different to that of the isothermal case. The type of bifurcation between the 100x and 10x cases has changed, and it appears there is a transition from a stable case with three fingers, to the more unstable case with two. Even though the two finger bifurcation in the isothermal case can be considered the 'stable' displacement at $Ca_g = 4000$, the smooth temperature variation from hot-cold around the interface tends towards a three finger bifurcation. The numerical scheme becomes much more accurate at these reduced Peclet numbers, meaning the three finger bifurcation is much easier to resolve.

The three finger bifurcation result therefore seems somewhat counter intuitive, but can be explained by the macroscopic behaviour of the temperature field. At the highest Peclet numbers, the convective flow creates a regime where a two finger bifurcation occurs with side branching later in time. This is a more



Figure 10: Interface plots at t = 30 for different diffusivity values. Left hand plots are case 1, right hand plots are case 2. Diffusivity stated in sub captions are the base diffusivities. — base diffusivity, - - - 10x base diffusivity. $Ca_g = 4000$.

unstable bifurcation than a three finger bifurcation, since with side branching, four fingers will exist shortly after the first bifurcation. The increased diffusivity reduces this effect and the bifurcation drops to a three finger bifurcation. The Peclet number however is still high, so the regime is convection dominated. If the diffusivity were increased further, then the finger bifurcation would transition back to the two finger displacement, as the perturbed pressure has less and less effect.

Increasing the Peclet number increases the number of fingers at bifurcation in the hot-cold injection cases from $2 \rightarrow 3 \rightarrow 2$ (with side branching), which would likely increase to four full fingers if the Peclet number were increased significantly beyond $Pe > 2 \times 10^4$. In realistic fluid injection problems, the Peclet number is likely to be above unity for cases with liquids, since the thermal diffusivity is low. This means that the flow will be convection dominated (at least at early times) and the more unstable bifurcations much more likely to occur.



Figure 11: Interfaces at t = 30 overlaying contour plots of the temperature field. $Ca_g = 4000$.



Figure 12: Interface temperature scatter plots at t = 30 for different diffusivity cases. $Ca_g = 4000$.

5 Conclusion

In this work, we have formulated two new RBF-FC multi-zone methods for the solution of both steady-state and transient convection-diffusion transport type problems. The methods were coupled with an efficient BEM for the solution of immiscible thermo-viscous fingering.

Non-isothermal injection cases revealed that negative temperature gradients induced by hot fluid injection into a resident cold fluid, (with an unfavourable viscosity contrast) enhanced the destabilisation of the interface and promoted the onset of viscous fingering. The very high thermal Peclet number ($Pe > 1x10^4$) created a convective regime, which significantly destabilised the interface. In cases where a cold fluid was injected into a hot fluid, the interface could be stabilised, and bifurcation delayed. However, with a naturally unfavourable viscosity contrast between the fluids, bifurcation would always occur eventually as the temperature field diffused with time and the stabilisation effects became less prominent.

Decreasing the thermal Peclet number reduced the destabilising effects of hot-cold injection and reduced the stabilising effects of cold-hot injections. Even with 100x increases in diffusivity the thermal regime was still highly convective, tending to a bifurcation of three fingers as opposed to the two finger isothermal case. For the bifurcation to reach that of the isothermal case, the Peclet number would have to be reduced more significantly, to values less than unity.

6 Acknowledgements

The authors would like to thank the University of Nottingham HPC team for use of the Minerva cluster. The present work has been partially supported by an EPSRC post-graduate research scholarship.

References

- A. Shafiei and M.B. Dusseault. Geomechanics of thermal viscous oil production in sandstones. Journal of Petroleum Science and Engineering, 103:121 – 139, 2013.
- [2] Y. Wang, C. Zhang, N. Wei, M. Oostrom, T.W. Wietsma, X. Li, and A. Bonneville. Experimental study of crossover from capillary to viscous fingering for supercritical CO₂ - water displacement in a homeogneous pore network. *Environmental Science and Technology*, 47:212 – 218, 2013.
- [3] C.W. Park and G.M. Homsy. Two phase displacement in Hele-Shaw cells: Theory. Journal of Fluid Mechanics, 139:291 – 308, 1984.
- [4] D. Pritchard. The instability of thermal and fluid fronts during radial injection in a porous medium. Journal of Fluid Mechanics, 508:133 – 163, 2004.
- [5] T. Sheorey and K. Muralidhar. Isothermal and non-isothermal oil-water flow and viscous fingering in a porous medium. *International Journal of Thermal Sciences*, 42:665 – 676, 2003.
- [6] D. Pritchard. The linear stability of double-diffusive miscible rectilinear displacements in a Hele-Shaw cell. Eureopean Journal of Mechanics B/Fluids, 28:564 – 577, 2009.
- [7] M.C. Kim. Onset of radial viscous fingering in a Hele-Shaw cell. Korean Journal of Chemical Engineering, 29(12):1688 – 1694, 2012.
- [8] M. Mishra, P.M.J Trevelyan, C. Almarcha, and A. De Wit. Influence of double diffusive effects on miscible viscous fingering. *Physical Review Letters*, 105(204501), 2010.
- M.N. Islam and J. Azaiez. Miscible thermo-viscous fingering instability in porous media. Part 1: Linear stability analysis. *Transport in Porous Media*, 84:821 – 844, 2010.
- [10] S.J. Jackson, D. R. Stevens, H. Power, and D. Giddings. A boundary element method for the solution of finite mobility ratio immiscible displacement in a Hele-Shaw cell. *International Journal for Numerical Methods in Fluids*, 78(9):521 – 551, 2015.
- [11] D. Stevens, H. Power, C.Y. Meng, and D. Howard. An alternative local collocation strategy for highconvergence meshless PDE solutions, using radial basis functions. *Journal of Computational Physics*, 245:52 – 75, 2013.
- [12] H. Power and L.C. Wrobel. Boundary Integral Methods in Fluid Mechanics. Computational Mechanics Publications, Southampton, 1995.
- [13] J.J.S.P Cabral, L.C. Wrobel, and C.A. Brebbia. A BEM formulation using B-splines: I Uniform blending functions. *Engineering Analysis with Boundary Elements*, 7(3):136–144, 1990.
- [14] S.J. Jackson, D. Stevens, D. Giddings, and H. Power. An adaptive RBF finite collocation approach to track transport processes across moving fronts. *Computers and Mathematics with Applications*, 71(1):278 – 300, 2016.
- [15] D. Stevens, H. Power, M. Lees, and H. Morvan. A local hermitian RBF meshless numerical method for the solution of multi-zone problems. *Numerical Methods for Partial Differential Equations*, 27(5):1201 – 1230, 2011.
- [16] H. Chen, C. Min, and F. Gibou. A supra-convergent finite difference scheme for the poisson and heat equations on irregular domains and non-graded adaptive cartesian grids. *Journal of Scientific Computing*, 31:19 – 60, 2007.
- [17] H. Chen, C. Min, and F. Gibou. A numerical scheme for the stefan problem on adaptive cartesian grids with supralinear convergence rate. *Journal of Computational Physics*, 228:5803 – 5818, 2009.
- [18] T.D. Aslam. A partial differential equation approach to mutilidimensional extrapolation. Journal of Computational Physics, 193:349 – 355, 2003.
- [19] G.M. Homsy. Viscous fingering in porous media. Annual Review of Fluid Mechanics, 19:271–311, 1987.