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An Implicit Meshless RBF-based Differential Quadrature Method Applied to the Lid-Driven Cavity Problem

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Abstract: In the present study, a numerical investigation of steady-state Stokes flow problem on a lid-driven square cavity is carried out using mesh-free local radial basis function-based differential quadrature (RBF-DQ) method. This method is a combination of differential quadrature approximation of derivatives and function approximation of RBF. The weighting coefficients of (RBF-DQ) method are determined by using Radial Basis Functions (RBF) as test functions instead of using high-order polynomials. Discretized derivatives of velocity and pressure at a point is defined by a weighted linear sum of functional values at its neighboring points. In this work, this method is applied to the two-dimensional Stokes flow in a fully coupled form using a staggered arrangement of primitive variables. Results obtained from the RBF-DQ method are compared with the existing result in the literature on lid-driven cavity problem. In order to get better understanding for the RBF-DQ method, outcomes are discussed in details.

Keywords: Meshless Methods, Differential Quadrature, Radial Basis Functions, Numerical Algorithms, Computational Fluid Dynamics.

1 Introduction

Stokes flow problems in which the viscous effects dominates the inertial and gravitional effects are observed in many practical applications such as MEMS, polymer/food manufacturing, settling of dust particles and the swimming of microorganisms. Traditional numerical techniques such as finite difference (FDM), finite volume (FVM) and finite element methods (FEM) are used to solve many different complex problems including Stokes flow. However, it is very well known that these methods strongly rely on the mesh properties and for complex domains, mesh generation takes considerable amount of time. In order to overcome mesh-related difficulties, mesh-free methods have been developed.

Recently, a new mesh-free method was proposed based on the so-called radial basis functions (RBF)[1, 2]. Especially for higher dimensions, it is found that RBFs are able to construct an interpolation scheme with important properties such as higher efficiency, good quality and capability of handling scattered data [3]. In order to approximate derivatives by using RBFs, the RBF-DQ method, which combines the differential quadrature approximation of derivatives and function approximation of RBF is proposed by Shu and co-workers [3, 4].

In the present study, RBF-DQ method is applied to a two-dimensional domain of lid-driven cavity problem for Stokes flow. In the proposed approach, mass and momentum equations are solved in a coupled manner. Obtained results are in good agreement with the available data in the literature.

2 Local MQ-DQ Method

The essence of the DQ method is that the partial derivative of an unknown function with respect to an independent variable can be approximated by a linear weighted sum of functional values at all mesh points. Assuming that a function $f(x)$ is sufficiently smooth, its *m-th* order derivative with respect to x at a point x_i can be approximated by DQ as [4]

$$\frac{\partial^m f(x_i)}{\partial x^m} = \sum_{j=1}^{N_s} w_{ij}^m f(x_j) \quad i = 1, 2, 3, ..N \tag{1}$$

where x_j, $j = 1, 2, 3, ..N_s$(number of supports) are the discrete support nodes of x_i, $f(x_j)$ and w_{ij}^m are the function values at these points and the related weighting coefficients, respectively. The index i refers to the reference node in a global discretization of N nodes while j is a local index for the respective support nodes as shown in Figure 1. The key of Local DQ method is the determination of weighting coefficients

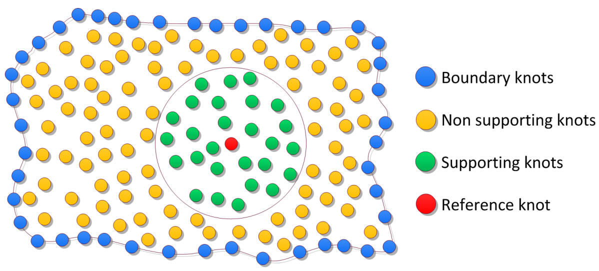


Figure 1: Supporting knots around a specific knot

w_{ij}^m which requires a set of N_s basis functions. This approach can be naturally applied to any dimensions. One of the important advantages of the DQ method as a global (high-order) approach, in comparison with the low order methods such as FVM or FDM, is its ability to generate numerical results with high order of accuracy by using a considerably small number of mesh points. Among various RBFs, the multiquadric function (MQ) is chosen due its to accuracy, stability and efficiency [1]. Local DQ with Multiquadrics (MQ) RBF ($\phi(r) = \sqrt{r^2 + \epsilon^2} \quad \epsilon > 0$ where r$=\|x - x_j\|_2$) is known as Local MQ-DQ.

In order to approximate *m-th* order derivative with respect to x of a function $f(x)$ at a point x_i, we obtain the following system of linear algebraic equations by substituting RBF in Equation (1) for the weighting coefficients

$$\frac{\partial^m \phi_p(x_i)}{\partial x^m} = \sum_{j=1}^{N_s} w_{ij}^{(m)} \phi_p(x_j) \tag{2}$$

where $\phi_p(x) = \phi(x, x_j)$ [5]. In matrix form, the vector of weighting coefficients w_{ij} for the first order derivative can be given as

$$\begin{bmatrix} \dfrac{\partial \phi_1(x_i)}{\partial x} \\ \dfrac{\partial \phi_2(x_i)}{\partial x} \\ \vdots \\ \dfrac{\partial \phi_N(x_i)}{\partial x} \end{bmatrix} = \begin{bmatrix} \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_N) \\ \phi_2(x_1) & \phi_2(x_2) & \cdots & \phi_2(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(x_1) & \phi_N(x_2) & \cdots & \phi_N(x_N) \end{bmatrix} \begin{bmatrix} w_{i1}^{(1)} \\ w_{i2}^{(1)} \\ \vdots \\ w_{iN}^{(1)} \end{bmatrix} \tag{3}$$

Relative error is dependent to the the shape parameter and the stencils. Figure 2 shows different stencils which can be used in calculations. For this study, a circular region is chosen as illustrated in Figure 1. However, for each reference knot, the number of supporting knots may be different.

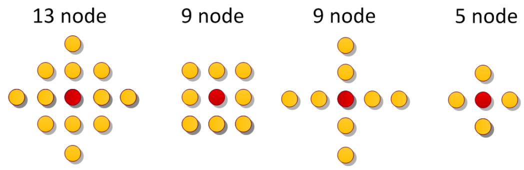


Figure 2: Different supporting knots distributions

In the Local MQ-DQ method, the shape parameter ϵ may have a strong influence on the accuracy of numerical results. The number of supporting knots and the size of supporting region mainly effect the optimal value of ϵ. The size effect of supporting region can be minimized by normalization of scale in the supporting domain. The following transformation can be made for normalization to transform the local support region to a unit circle for the two dimensional case [4]

$$\bar{x} = \frac{x}{R_i}, \quad \bar{y} = \frac{y}{R_i} \tag{4}$$

where (x, y) represents the coordinates of supporting region in the physical space, $(\bar{x}, \bar{y})$ denotes the coordinates in the square, R_i is the minimal radius of the circle for the chosen knot i, enclosing all knots in the supporting region. The coordinate transformation in Equation (4) also changes the formulation of the weighting coefficients. For example, by using the differential chain rule, the first order partial derivative with respect to x in the Local MQ-DQ approximation can be written as

$$\frac{\partial \phi_p(x)}{\partial x} = \frac{\partial \phi_p(x)}{\partial \bar{x}} \frac{d\bar{x}}{dx} = \frac{\partial \phi_p(x)}{\partial \bar{x}} \frac{1}{R_i} = \sum_{j=1}^{N_s} \frac{w_{ij}^{(1)}}{R_i} \phi_p(x_j) \quad i = 1, 2, 3, ..N \tag{5}$$

It can be seen that the shape parameter ϵ is equivalent to $R_i \bar{\epsilon}$ due to scaling. Thus, the modified RBF can be given as following

$$\phi(r) = \sqrt{r^2 + \epsilon^2} = \sqrt{(x - x_i)^2 + (y - y_i)^2 + \epsilon^2} = \sqrt{(\bar{x} - \frac{x_i}{R_i})^2 + (\bar{y} - \frac{y_i}{R_i})^2 + \bar{\epsilon}^2} \tag{6}$$

3 Problem Statement

For the incompressible viscous fluid flow in the Cartesian coordinate system, the continuity and the momentum equations with no convective terms (due to small Reynolds number, meaning that viscous forces dominate the dynamics) represent the Stokes flow. The continuity, x-momentum and y-momentum equations can be written as follows:

$$-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{7}$$

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{8}$$

$$\frac{\partial p}{\partial y} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \tag{9}$$

The boundary conditions for a square domain $[0,1]^2$ are given as

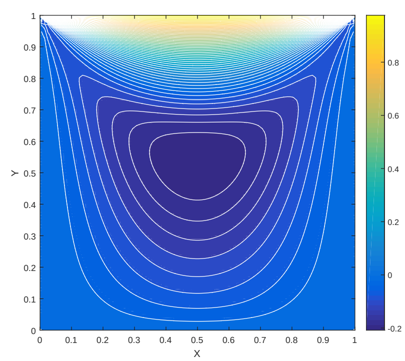
$$u(0,y)=0 \quad v(0,y)=0 \tag{10}$$
$$u(1,y)=0 \quad v(1,y)=0 \tag{11}$$
$$u(x,0)=0 \quad v(x,0)=0 \tag{12}$$
$$u(x,1)=1 \quad v(x,1)=0 \tag{13}$$

The following steps describe the solution procedure:

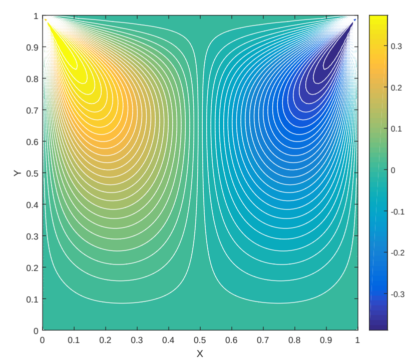
1. Create uniformly distributed nodes for velocity and pressure variables.
2. Specify the shape parameter needed for MQ RBF.
3. Specify the radius of the circle which contains the supporting nodes around the specific node.
4. Calculate the weights for x and y derivatives of RBF (first and second derivatives).
5. Convert the PDEs in to the algebric equations ($Ax=b$) using calculated weights and apply the boundary conditions in order to solve the primitive variables as follows:

$$\begin{bmatrix} x-mommentum \\ y-mommentum \\ continuity \end{bmatrix} \Rightarrow \begin{bmatrix} [\nabla^2] & 0 & [\partial/\partial x] \\ 0 & [\nabla^2] & [\partial/\partial y] \\ [\partial/\partial x] & [\partial/\partial y] & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{14}$$

6. Plot the results and compare with the benchmark work.



(a)



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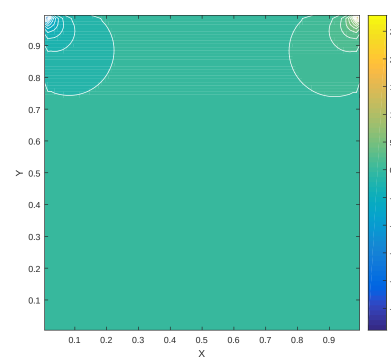

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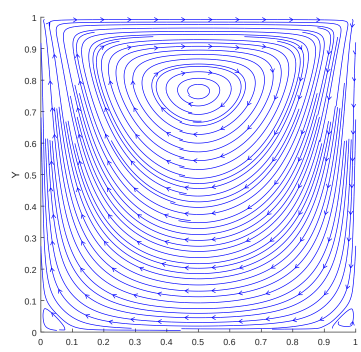

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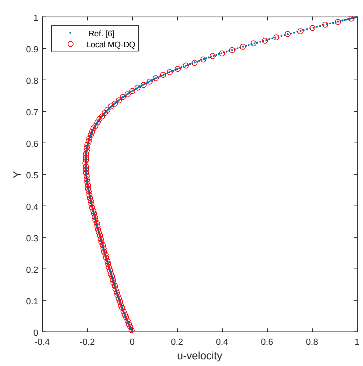

(e)

Figure 3: a) u-velocity b) v-velocity c) Pressure d) Streamlines (using Local MQ-DQ for $\bar{\epsilon}^2 = 1$, 10000 pressure and 22000 velocity points) e) u-velocity from Ref. [6] and Local MQ-DQ at $x = 0.5$

Table 1: Results for Local MQ-DQ

$\bar{\epsilon}^2$	Velocity nodes	Pressure Nodes	RMS Error	Maximum Error
0.05	220	100	6.548486e-03	1.568317e-02
1.0	220	100	6.901535e-03	1.621041e-02
0.05	5100	2500	7.973957e-04	2.532979e-03
1.0	5100	2500	6.865774e-04	2.089763e-03
0.05	20200	10000	3.980599e-04	1.312594e-03
1.0	20200	10000	3.309334e-04	1.076999e-03

The number of points for velocity/pressure variables and scaled shape parameter square value $\bar{\epsilon}^2$ used in the numerical simulations are given in Table 1. The nodes are distributed uniformly in the domain. For calculations 0.05 and 1.0 values are chosen for the scaled shape parameter square $\bar{\epsilon}^2$. Local MQ-DQ results for $\bar{\epsilon}^2 = 1$ and 10000 pressure, 20200 velocity nodes are presented in Figure 3. The values taken from Ref. [6] for horizontal u velocity is compared with three different node distribution (10, 50 and 100) at $x = 0.5$ in terms of RMS and Maximum error in Table 1. It was found that the numerical results obtained by Local MQ-DQ are in good agreement with the one given in Ref. [6] as shown in Figure 3e and Table 1. RMS and Maximum error decrease with increasing number of points. Moreover, selected values of shape parameter ϵ have a limited effect on the RMS ($\%5 - \%17$) and Maximum ($\%3 - \%18$) errors.

4 Conclusion and Future Work

Two dimensional Stokes flow in a lid-driven cavity is analyzed numerically by implicit mesh-free Local MQ-DQ method. It is found that the numerical results obtained by the Local MQ-DQ method agree very well with available data in the literature. The key importance of this study is solving the continuity and the momentum equations in fully coupled form for Local MQ-DQ method, which, to the authors' knowledge, has not been presented in the literature before. The proposed method can also be extended for the applications of steady and unsteady flows at different Reynolds numbers in two or three dimensions.

References

[1] R Franke. Scattered data interpolation - tests of some methods. *Mathematics of Computation*, 38(157):181–200, 1982.

[2] EJ Kansa. Multiquadrics - A scattered data approximation scheme with applications to computational fluid-dynamics .2. solutions to parabolic, hyperbolic and elliptic partial-differential equations. *Computers & Mathematics with Applications*, 19(8-9):147–161, 1990.

[3] H Ding, C Shu, and DB Tang. Error estimates of local multiquadric-based differential quadrature (LMQDQ) method through numerical experiments. *International Journal for Numerical Methods in Engineering*, 63(11):1513–1529, 2005.

[4] C Shu, H Ding, and KS Yeo. Local radial basis function-based differential quadrature method and its application to solve two-dimensional incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 192(7-8):941–954, 2003.

[5] Wu Y.L. and Shu C. Development of RBF-DQ method for derivative approximation and its application to simulate natural convection in concentric annuli. *Computational Mechanics*, 29(6):477–485, 2002.

[6] M. Sahin and R. G. Owens. A novel fully implicit finite volume method applied to the lid-driven cavity problem-Part I: High Reynolds number flow calculations. *International Journal for Numerical Methods in Fluids*, 42(1):57–77, 2003.