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Numerical Analysis of Multiplicity and Transition Phenomena in Natural Convection

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Abstract: This research is devoted to the numerical investigation of the transition and multiplicity phenomena in natural convection within 2-D cavity. In this study the effects of different parameters such as Prandtl number (Pr), Rayleigh number (Ra), and aspect ratio (AR) are numerically analyzed. Transition and multiple convective flow patterns are observed using Rayleigh, boundary condition and pseudo-transient continuation methods. In boundary condition continuation method, the various temperature continuation methods are used as the initial values on boundaries. Combination of continuation methods result in observing different convective flow patterns.

Keywords: Transition and multiplicity phenomena, natural convection, continuation methods, Rayleigh number, Prandtl number, aspect ratio.

1 Introduction

In recent years, natural convection received considerable interest in various fields such as new electronic devices, solar thermal receiver systems, solidification process and biomedical. The natural convection problem reveals a variety of complex behaviors and high sensitivity to small difference of parameters such as the aspect ratio (AR), Rayleigh number (Ra), Prandtl number (Pr), and the thermal boundary conditions (TBC). The large body of literature available on natural convection was organized according to the parametric investigation to observe oscillatory and bifurcation phenomena.

At specific Ra number (bifurcation point), the transition from conduction (Nu=1) to convection (onset of convective instability) occurs. The first observations of transition and unstable natural convection phenomenon happened in experimental researches [1, 2]. The unstable natural convection has been specifically studied in some researches [3, 4, 5] for two dimensional (2D) and three dimensional (3D) cavities. However, these studies were performed only for variation of Pr numbers. The onset of unstable natural convection has been analyzed in Gelfagt *et al.* studies [6, 7]. Their studies focus on onset of instability phenomena in convective fluid flow for different AR. They presented the stability diagram for different Ra numbers and aspect ratio. They proposed that the Prandtl number and the aspect ratio play the significant role in changing steady to transient natural convection and multiple convective flow patterns.

The severe effects of natural convection on melt flow have motivated recent studies to examine convective flow [8]. Natural convection in melt flow affects the process of crystallization, and it completely affects the interface and crystal formation. It strongly changes the solute and temperature distribution within the melt flow [2].

A recent approach which studied the natural convection by using ISPH method was introduced by Danis *et al* [9]. ISPH method is used to simulate transient and laminar natural convection in a square cavity with Boussinesq approximation in specific rang of Rayleigh numbers 10^3 and 10^6. Despite the fact that the SPH is always used to discretize the Lagrangian equations, in this study the uniform Eulerian grids is discretized by using SPH operators. In other meshless Lagrangian method the SPH particles are moving inside the domain, but in this approach SPH particles are kept stationary. Since all particles are stationary, the Eulerian form of governing equations is used instead of the Lagrangian form of governing equations. In addition, they used an incompressible approach which is called Incompressible SPH (ISPH). Moreover, ISPH method was used to prevent the density error accumulation and particle disordering. In this method, the incompressibility is directly imposed an intermediate velocity field which is obtained without considering gradient of pressure.

Multiple convective flow patterns were investigated by Puigjaner et al. [10]. All stable and unstable flow patterns for Rayleigh-Benard convection problem with Pr=0.7 at different Ra numbers were analyzed by using parameter continuation techniques. Multiple convection flow patterns were observed even for moderate Ra numbers. The different patterns were discussed by streamlines direction and number of convection rolls and its different shapes.

Stochastic analysis by using random initial values for Rayleigh number is performed by D. Ventury et al. [11]. In his study, deterministic analysis was performed to capture steady-state solutions and primary bifurcations. In steady states problem the multiple stable solution found within specific ranges of Rayleigh number. Finally, the stochastic analysis were carried out random initial condition flows around bifurcation points to analysis the transient natural convection behavior.

Sheu et al. [12] discussed the transient convection and the onset of bifurcation point. They have studied the multiplicity phenomena in a three dimensional (3D) model which is numerically demonstrated that a unique symmetric and steady-state solution exist in small Rayleigh number (Ra). As the Rayleigh number was increased beyond onset of bifurcation point, the symmetric results became asymmetric, and from one point (bifurcation point) obtained two different solutions.

Various numerical studies have been performed on the instability of steady and transient natural convective flows. Mercader *et al.* [13] studied the parametric variation, BC, and periodic analysis. Firstly, they analyzed the basic state and its primary bifurcation that has two cases of temperature BCs, which are performed in horizontal plates with Pr= 0.007 with rectangular cavity of AR=2. Furthermore, periodic analysis was considered in their research, in which they studied both steady and transient convective flow. Finally, it was observed that by increasing thermal effects, secondary bifurcations revealed for both temperature profile cases.

The oscillatory convection within the liquid phase of two-phase flow was observed [1, 2], and some numerical and experimental studies[6, 14] investigated the oscillatory natural convection. Although these studies identify the phenomenon, a complete analysis that can cover the bifurcation points and multiple results is still lacking. In addition, It was mentioned that at critical Ra number, the transition and oscillatory natural convection intensively affect the fluid flow within closed domain. Joo-Sik Yoo [15] studied the combined effect of thermal and hydrodynamic instability natural convection in a narrow horizontal concentric annulus with Pr=0.4. The multiple natural convection patterns were shown in his investigation within annular gap between two concentric cylinder. The results represented the complicated multicellular flow. Increasing Ra number changes the multicellular flow to a nearly monocellular structure. Periodic steady solutions were observed within higher range of Ra numbers.

According to the literature, it is still uncertain why crystal growth of some materials is easy and for others it is not [14, 16]. It has been neglected that at the low Pr numbers ($Pr < 0.1$), the oscillatory natural convection flow occurs at the range of Ra numbers which are so close to the experimental conditions [16]. The strength of fluid flow and the unstable transition natural convection can strongly change the interface of two-phase flow. The effects of multiplicity and transition of natural convection has been neglected to estimate the shape and movement of the solid/liquid interface of two-phase flow [14]. The effects of BC and specially temperature gradient are investigated by Erenburg et al. [17]. Partially heated walls are set up as boundary condition to analyze the multiplicity and bifurcations of natural convection. In this research, both continuous and partial temperature gradient on wall are considered. Selver et al.[14] studied the partial heated vertical walls to simulate the floating-zone crystal growth.

Most of the studies, which discuss oscillatory convection, were considered the thermocapillary effects for upper BC on fluid flow[18]. The buoyancy force is the dominant force to carry out the natural convection in closed domain; however, the influence of thermocapillary forces on buoyancy-driven convection cannot be ignored. This influences were numerically studied for open cavities with differentially heated walls[19]. In these studies the Reynolds number (Re) is also important because they used the Marangoni stress to solve the stress-free (open) BC. Thermocapillary can impose intensive effects on buoyancy-driven flow. The results of combination of these two forces can completely change the stability of natural convection within the domain.

Some studies used the thermal lattice Boltzmann method to study the natural convection [20, 21, 22]. In this method, instead of using finite difference, finite element and finite volume methods to solve Naiver - Stokes equations, the lattice Boltzmann method were employed. In addition, there are some new combinations of lattice Boltzmann method with other methods to create new explicit method [23], which is based on the lattice Boltzmann method (LBM) combined with Taylor series expansion and the least squares approach.

Most of natural convection studies were used Boussinesq approximation to simulate the convective flow. However, there are several studies which used non-Boussinesq assumption to model convective flow, Hamimid et al.[24] was used the time dependent Naiver - Stokes equations under the Low Mach Number approximation (LMN method). This investigation demonstrated that for large temperature differences, LMN compressible method obtained better results for convective flow. In addition, Vekstein (2004) [25] investigated natural convection without using Boussinesq approximation, considered the energy of liquid and gas to investigate the onset of convective instability. In this study, the onset of natural convection instability is discussed by energy of the fluid. In other words, a gravitational energy sustains the fluid flow in natural convection by interchanging a hotter fluid with less density to cooler one with more density. The distinguish convection instability between fluid and ideal gas was also discussed in this research. Unlike a gas, a liquid may be considered as almost incompressible fluid. This means that its density, in the general case, is a function of pressure and temperature $\rho = \rho(P, T)$. But since it has a very weak dependence on the pressure, one may simply consider $\rho = \rho(T)$.

Szewc et al. [26] discussed the Boussinesq approximation failure by using SPH method, in which natural convection in a square cavity with a Boussinesq and a non-Boussinesq formulation was studied. In significant differences of density due to temperature gradient, the dimensionless Gay-Lussac number is suggested to measure density gradient in non-isothermal flows. The effect of Gay-Lussac number was investigated for velocity field and Nusselt number of non-Boussinesq convective flow on their study.

When both heat and mass transfer affect natural convection, the double-diffusive convective flow is defined to solve the problem [27]. For multi-component mixing flows, the transport of enthalpy, due to species diffusion, can have a significant effect on the enthalpy field and should not be neglected. Lewis number ($Le = \frac{k}{\rho c_p D}$) is very important in these kinds of investigation; when the Le number for any species $Le \gg 1$ increase, the thermo-solutal effects become so significant to simulate the convective flow[28].

The effects of rotating flow on oscillatory natural convection was studied experimentally and numerically [29, 30, 31] . For specific range of Gr numbers, thermal oscillations are detected at several rotation rate. In addition, it is found that the frequency of oscillation is a function of the Gr number.

Different parametric studies were carried on to investigate the natural convection and find the bifurcation points [6, 32]. Although these studies identify the phenomenon, a study that can cover all parameters which are affect the transition natural convection is still lacking.

2 Problem Statement

The fluid motion with temperature effects results in governing equations in which natural convection plays a major role by buoyancy forces, and the buoyancy forces sustain the fluid flow. The fluid is assumed Newtonian and quasi-incompressible (Boussinesq approximation), and the Navier-Stokes equations coupled with the energy equation govern the flow with the constant physical properties except in the buoyancy term where ρ is taken as a linear function of the temperature. Despite the fact that there is a density differences within the domain, by the Boussinesq approximation , the density is assumed to be constant except for the body forces. The dimensionless form of equations for conservation of mass, momentum, and energy can be written as following [23, 33]:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{1}$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + Pr(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}) \tag{2}$$

$$\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + Pr(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}) + RaPrT^* \tag{3}$$

$$u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*} = (\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}) \tag{4}$$

In the above equations $u^*, v^*, t^*, p^* and T^*$ represent the dimensionless velocity, time, pressure and temperature, respectively, and the following non-dimensionalizations were defined $x^* = x/L; y^* = y/L;$ $u^* = uL/\alpha; v^* = vL/\alpha;$ $t^* = \frac{t\alpha}{L}; p^* = pL^2/\rho\alpha^2; T^* = (T - T_0)/\Delta T; Pr = \nu/\alpha$ and $Ra = g\beta_T \Delta T L^3/\nu\alpha$, where t, p, T, ν, α and β_T denote the dimensional time, dimensional pressure, dimensional temperature, viscosity, thermal diffusivity and the coefficient of thermal expansion, respectively. The importance of buoyancy forces in a mixed convection flow can be measured by the ratio of the Grashof and Reynolds numbers.

$$\frac{Gr}{Re^2} = \frac{g\beta\Delta TL}{\nu^2} \tag{5}$$

When this ratio exceeds unity, the buoyancy effect is dominant, and the natural convection is occurred. Conversely, if it is very small, buoyancy forces is weak and can be ignored. In pure natural convection, the strength of the buoyancy-induced flow is measured by the Ra number.

Rayleigh-Benard convection is an example of thermal instability where temperature difference between the top and bottom caused by heating the fluid from below results in formation of rolls. If the temperature gradient, density gradient, was large enough, the gravitational forces will dominate and instability will occur. Rayleigh-Benard instability has been a topic for many experimental and numerical studies [34]. The Rayleigh-Benard instability develops when Ra number is above a critical value. The effect of domain geometry is analysed by AR, which is actually the ratio of height to length (AR=H/L). Different aspect ratio(AR=1/4, 1/2, 1, 2, and 4) are considered in this research. The problem is defined using constant temperature at the top and the bottom. Insolated BC is defined on sidewalls.

To examine the mesh sensitivity, the average mesh size, h, is defined to represent cell mesh size. Different rectangular meshes (h=1, 0.5, 0.25,...) are used to investigate the effect of mesh refinement. Simulations are performed to obtain velocity magnitude and temperature contours at critical Ra number and critical positions (half-width, half-height and diagonal lines). The mesh size of h=1 is selected as a coarse mesh and then in every refinement step it is divided by two as the finer mesh. Mesh sensitivity analysis is carried out in AR=2, $Ra = 1 \times 10^5$ and Pr=1 since these values are the most critical ones. The no-slip velocity boundary condition on walls results in zero velocity on walls. In addition, the velocity magnitude values are mostly constant for half-width line. Therefore, the half-height and diagonal lines of domain are selected to compare the velocity magnitude values.

Velocity magnitude at half-height and on diagonal line are plotted for different mesh sizes as shown in Figure 1 (a) and (b). By comparing the results, one can notice that the shape of velocity magnitudes are the same for different mesh sizes but the amplitude of them varies for the first three mesh sizes. There is no significant change in results when the mesh is refined further after h=0.25.

3 Results and Discussion

The results of transition and multiplicity phenomena on different cases are represented in two separate section. First part of the results is devoted to the parametric investigation of Rayleigh-Benard Instability to show the transition natural convection for different parameters. The continuation methods, which are used in this research, play fundamental role to follow the results path to show the transition phenomenon in this section. In second part, multiple convective flow is shown and compared with literatures. The BC continuation method helps to obtain multiple convective patterns. Indeed, the BC and Ra continuation

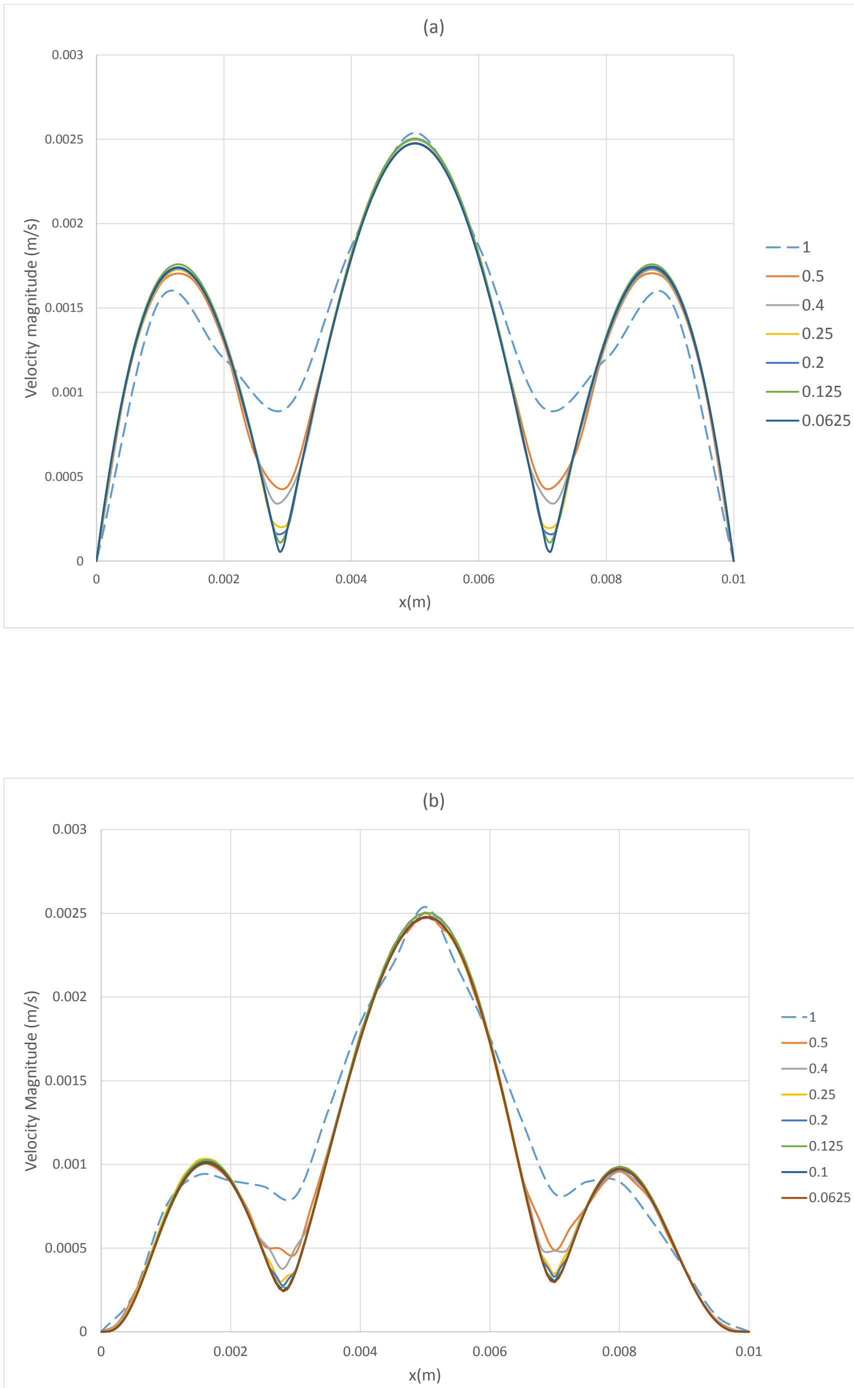

Figure 1: Velocity magnitude for AR=2, $Ra = 1 \times 10^5$ and Pr=1 for different mesh sizes; a) in x-direction at half-height; b) on the diagonal of the domain.

methods support to follow the various path of flow patterns after bifurcation points. The Ra continuation is a useful method to find critical Ra numbers and bifurcation points. At critical or high Ra numbers which the convergence of equation is not satisfied, continuation methods are used to obtain the stable results.

3.1 Parametric analyse and transition phenomenon

In this section, a parametric analyse for various Ra, Pr and AR is carried out. In every case, the BC continuation method is utilized as initial value on vertical walls. The Rayleigh-Benard instability develops when Ra number is above a critical value. Substantial number of cases were analyzed however five cases with AR=2 and $Ra = 2 \times 10^5$ were selected to represent the phenomenon as shown in Table 1. For all cases, the Nu number is bigger than 1 and convection heat transfer is dominant. By decreasing the Pr number from 10 to 0.1, the velocity magnitude, which is the mean value of velocity magnitude within the domain, is increased except for Pr=0.5 (Figure 2 (a)) in which the transition phenomenon is happened and two kidney shape velocity cells are changed to four velocity cells. In transition period between Pr=0.5 and Pr=1, the velocity magnitude is increased severely which is shown in Figure 3. It can be concluded that the critical Ra number increases with decreasing Pr number. Indeed, the bifurcation point for transition from two-rolls to four-rolls for low Pr numbers(lower than one) occurs at low Ra number values in these cases as shown in Figure 2. The higher velocity values are obtained for the cases which are in critical Ra number region.

Pr	Velocity magnitude	Nusselt
0.1	0.00222	2.23
0.5	0.00176	3.43
1	0.00325	8.07
5	0.000835	12.647
10	0.000418	12.665

Table 1: Parametric analysis for a range of Pr numbers between 0.1 and 10 at constant $Ra = 2 \times 10^5$ and $AR = 2$; maximum value of the Nu number and velocity magnitude is obtained for each case.

The results show that by increasing the Pr number, Nu number increase in a non-linear manner as shown in Figure 2 (b). The increase Nu can be described in two stages: at the first stage ($0.1 < Pr < 5$), it increases drastically. This region is indeed the transition region from two symmetric velocity cells to four symmetric velocity cells, and the slope of Nu-Pr graph is high. The second stage ($5 < Pr < 10$) is more like a plateau at which the increase in Nu number is negligible compared to the increase in Pr number.

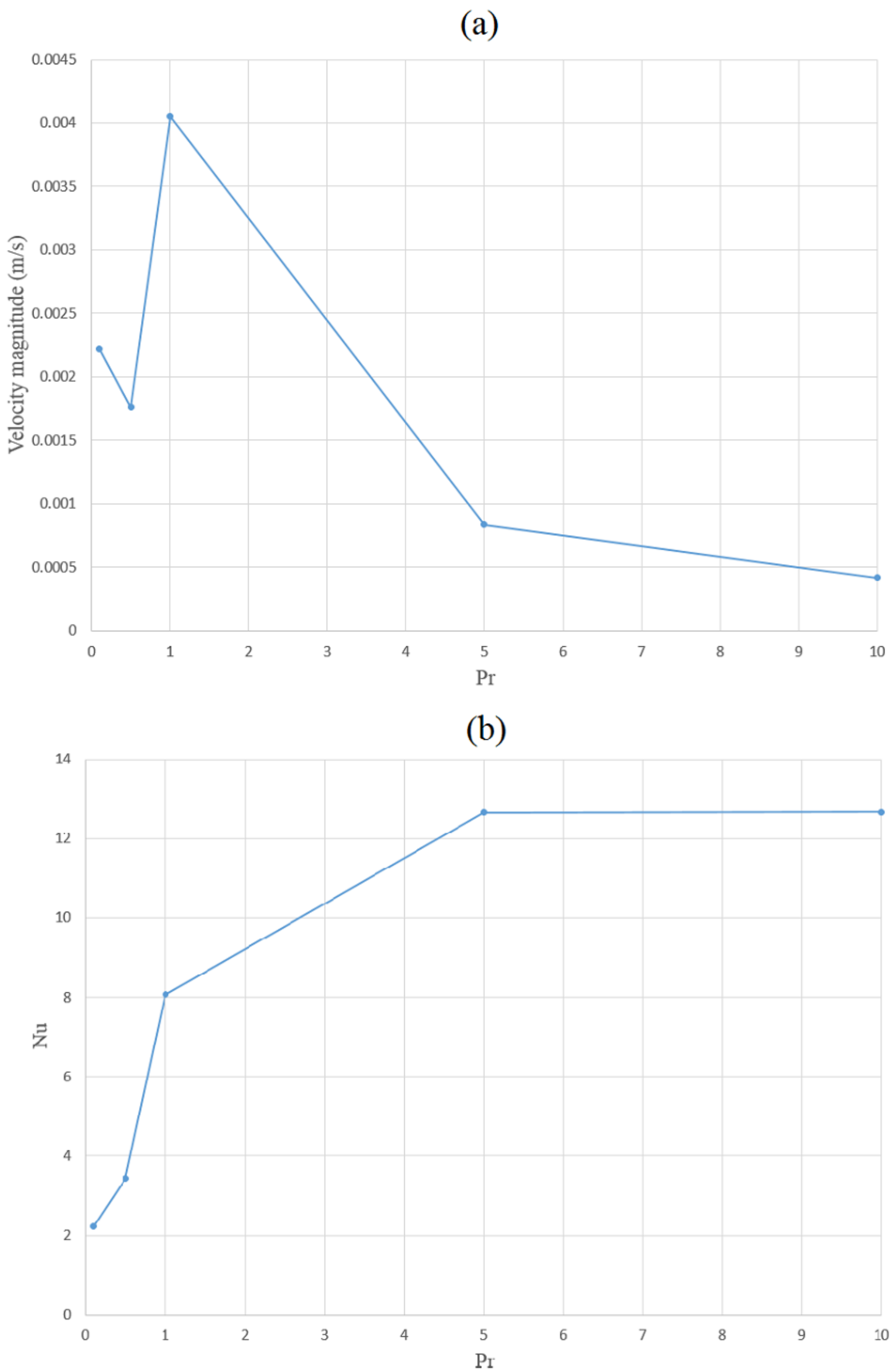

Figure 2: Plots showing the variation of velocity magnitude and Nu number with respect to constant Ra number and AR ($Ra = 2 \times 10^5$ and $AR = 2$); a) Pr-velocity magnitude; b) Pr-Nu.

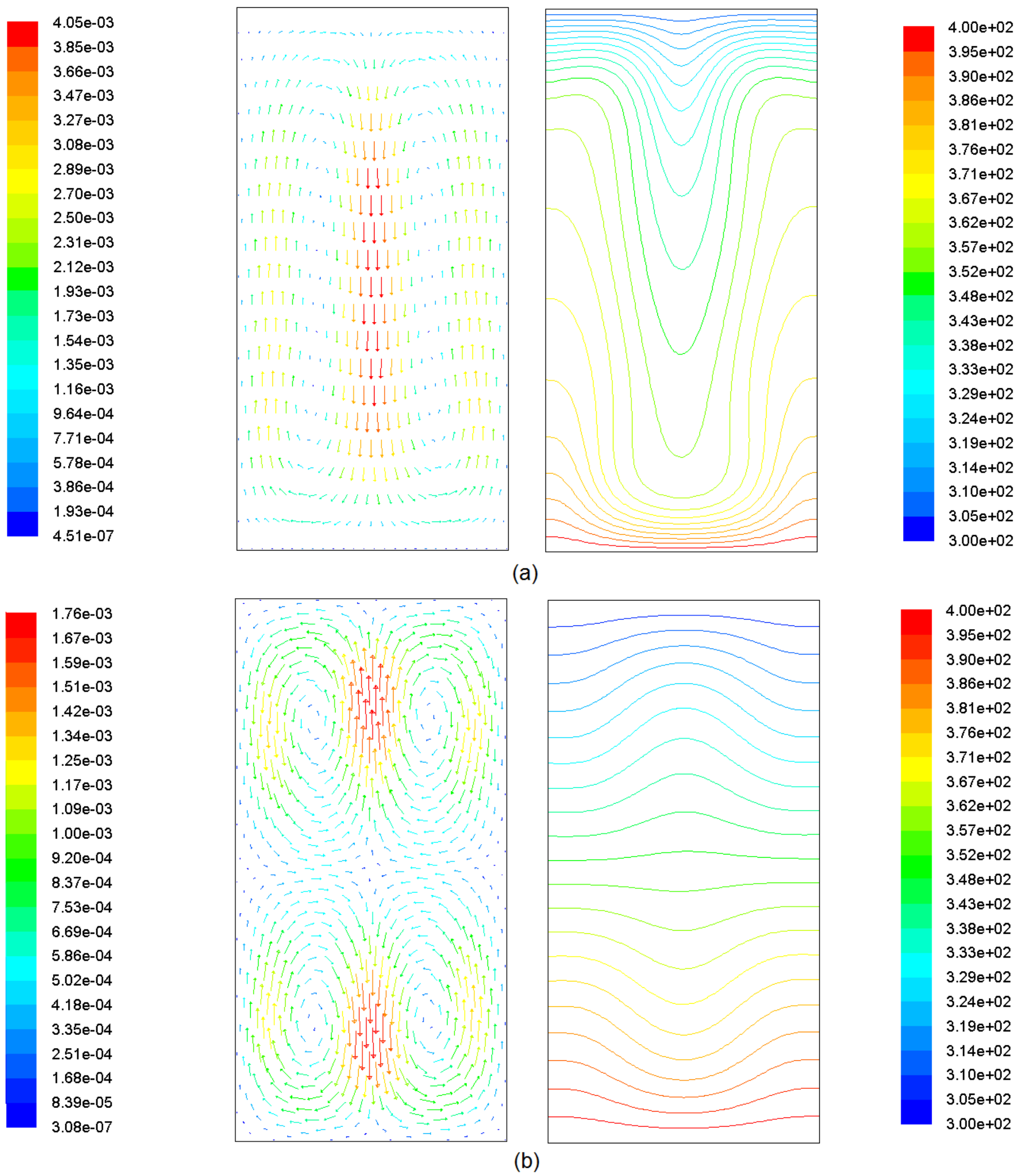

Figure 3: Temperature contours (right) and velocity vectors (left) at constant $Ra = 2 \times 10^5$ and $AR = 2$; a) Pr=1; b) Pr=0.5.

3.2 Multiplicity

A combination of thermal and hydrodynamic effects yield complex multiple flow patterns which are obtain the different patterns at the completely same problem setting. Although, Puigjaner et al. (2004) [10] and Venturi et al.(2010) [11] were investigated the multiple convective patterns. This study reperesent some important multiple patterns within the critical Ra number range which are not obtained in previous studies. The one-roll convection pattern was obtained by Venturi [11] in both directions (clockwise and counterclockwise) as shown in Figure 4(a). In present study, the transition phenomenon is observed at the range of Ra number between $Ra = 2500$ to $Ra = 3000$ and the one convection cell in both directions is shown in Figure 5 (a) and (b). The more accurate results are obtained because of continuation methods and better mesh generation. The two-cell convection patterns in two different direction are obtained in Figure 6. Venturi et al. [11] observed the secondary branch point as shown in Figure 4(b). This is in agreement with the results obtained in present study.

In this investigation, the third bifurcation point is happened at $Ra = 4 \times 10^4$ as shown in Figure 7 (a) and (b) in both directions. Two small cells appear at top corners (Figure 7 (a)) or at bottom corners (Figure 7 (b)). The three-cell result obtained by Venturi et al. [11] at Ra=21000 can be explained as an unstable result which is shown in Figure 4 (c). The results in the present study are stable as shown in Figure 7 (a) and (b). By increasing Ra to $Ra = 8 \times 10^4$, four symmetric stable cells appear as illustrated in Figure 7 (c). The probability of appearence of two small cells at the top or the bottom during the transition period are same. This observation can be considered as another form of multiplicity in convective flow which has not considered in previous studies.

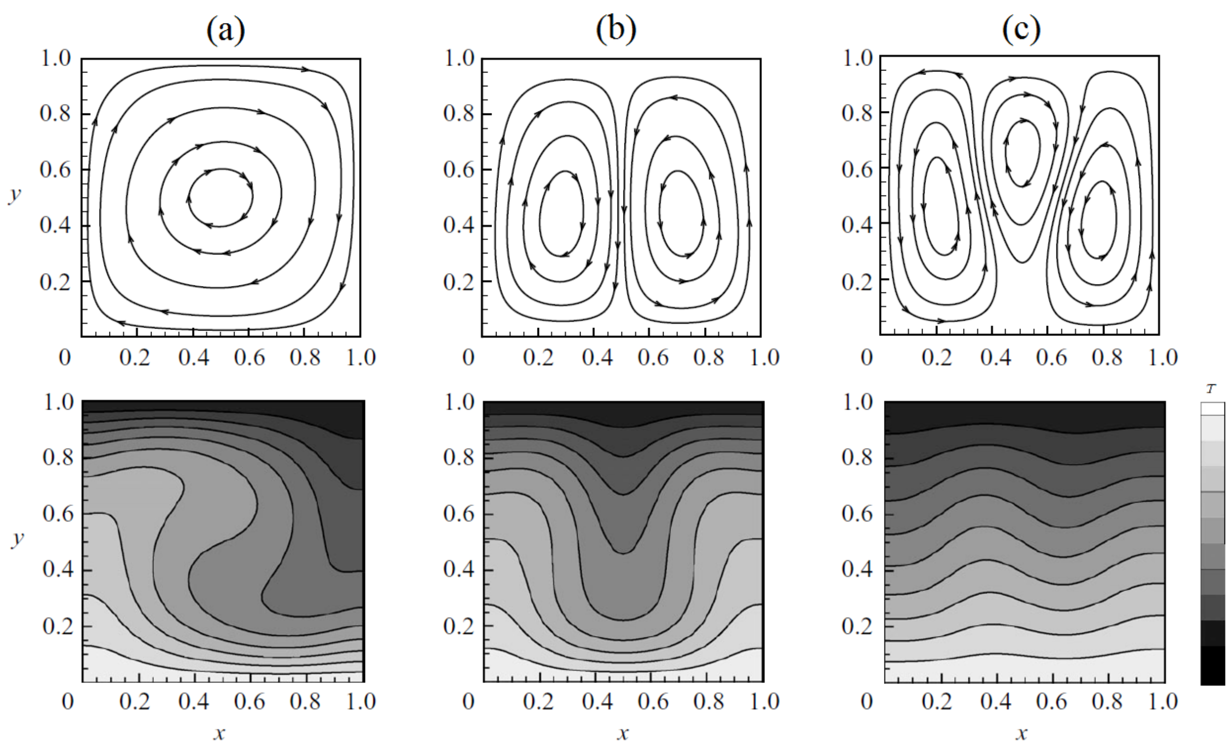

Figure 4: Velocity streamlines (first row) and temperature contours (second row) which are obtained by Venturi et al. (2010) [11]; a) one-roll convection pattern defined as clockwise roll and counterclockwise roll at Ra = 5000 ; b) two-roll at Ra=15000 ; c) three-roll convection pattern at Ra=21000.

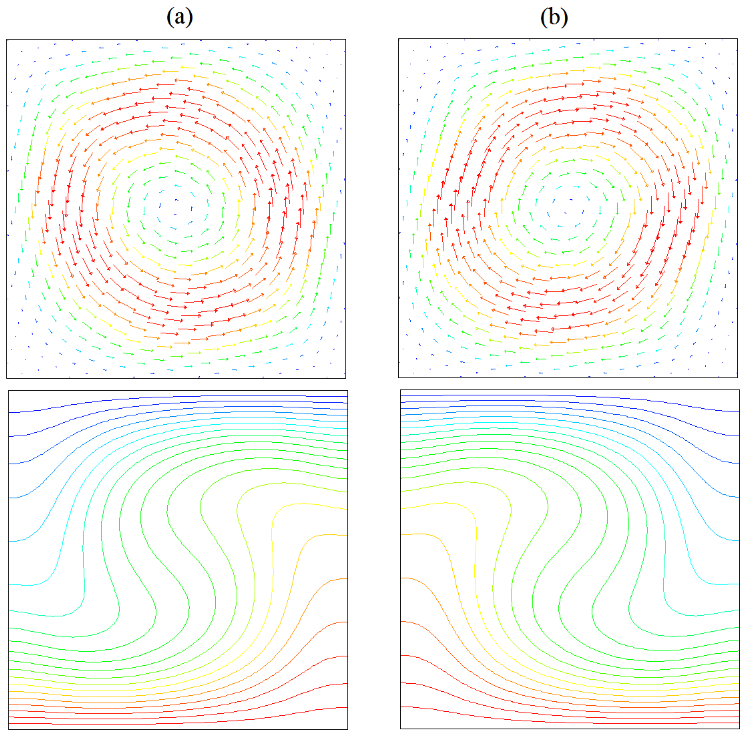

Figure 5: Velocity magnitude vectors and temperature contours AR=1, Ra=5000 and Pr=0.7 ; a) counter-clockwise ; b) clockwise .

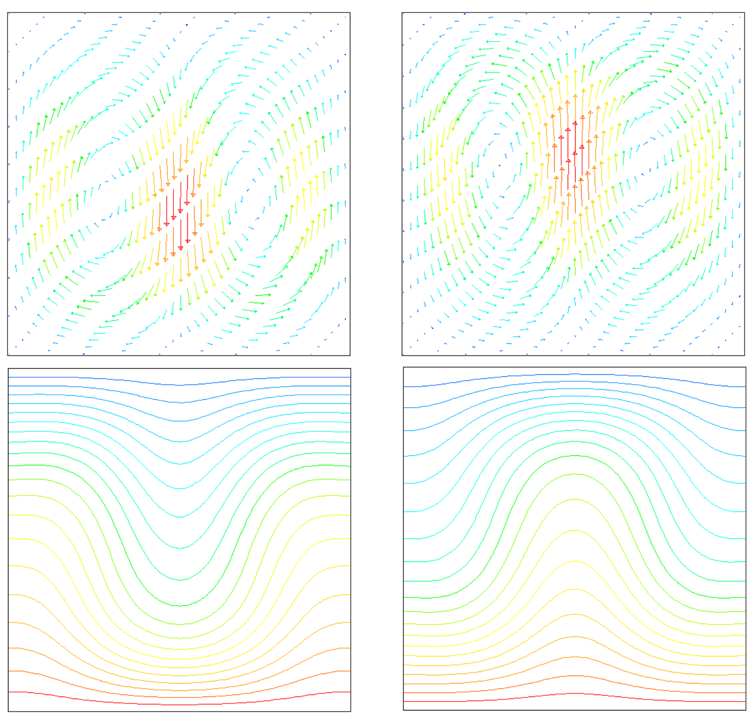

Figure 6: Velocity magnitude vectors and temperature contours for AR=1, $Ra = 1 \times 10^4$ and Pr=0.7; two-roll convection pattern in opposite directions.

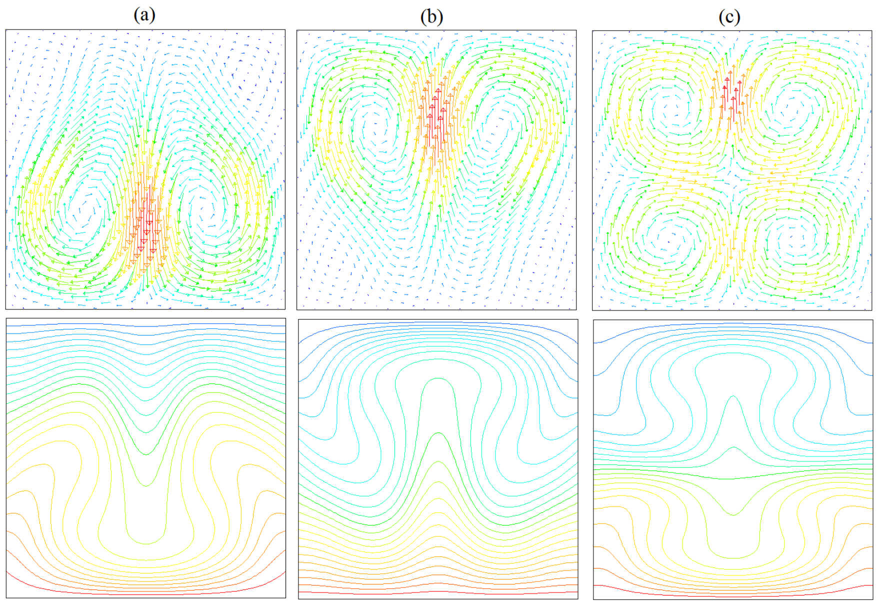

Figure 7: Velocity magnitude vectors and temperature contours; a) and b) four-roll stable patterns in opposite directions $Ra = 4 \times 10^4$; c) four symmetric stable cells at $Ra = 8 \times 10^4$.

Figure 8 shows the two unstable convection cells which represent a different flow pattern in transition period between two cells to four cells.

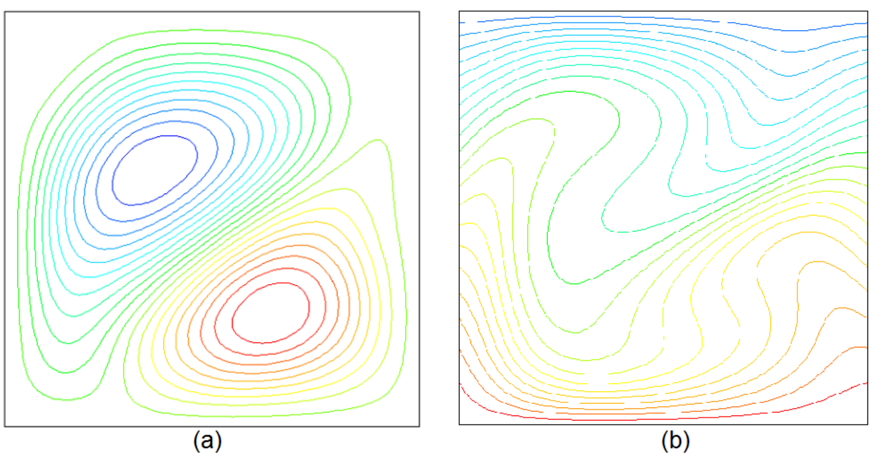

Figure 8: a) Unstable pattern of velocity stream-function; b) Temperature contours.

4 Conclusion and Future Work

In this study, the transition and multiple results phenomena for natural convection are investigated. The BC and Ra continuation methods are implemented to obtain transition and multiple results. A parametric analysis at different Pr numbers is carried out for constant $Ra = 2 \times 10^5$ and $AR = 2$. The outcome reveals a critical Ra numbers for which transition phenomenon occurs. According to the results, by decreasing the Pr number from 10 to 0.1, the velocity magnitude is increased except for Pr=0.5 in which the two kidney shape velocity cells are divided to four velocity cells that leads to a decrease in the velocity magnitude. It can also be concluded that the critical Ra number increases with decreasing Pr number. Indeed, the bifurcation point for transition from two-rolls to four-rolls for low Pr numbers(lower than one) occurs at low Ra number values in these cases as shown in Figure 3. In all of the cases, behavior of convective flow changes significantly within the transition region. This can be explained by energy balance within the enclosure as studied in experimental research by Mishra et al. [34].

Multiple convection flow patterns are obtained. The multiple results are presented in three form: the direction of velocity streamline vectors, the number of cells and shape of convection flow patterns. The transition between multiple results for the same parameters shows that one unstable result can be converted to other result.

The results in this research provide new areas of study for future works. A similar study can be investigated for the double-diffusive problem. The effects of transition and multiplicity phenomenon in mass concentration for pure flow or multi-component mixing flows can be considered as future work. In addition, considering the effects of transition natural convection on various new application such as new electronic devices, solar thermal receiver systems, solidification process, this study can be received considerable interest in various applications.

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