Ninth International Conference on Computational Fluid Dynamics (ICCFD9), Istanbul, Turkey, July 11-15, 2016 **ICCFD9-xxxx**

A New Computational Package for Using in CFD and Other Problems

Mohammad Reza Akhavan Khaleghi

The Office of Counseling and Research Fluid Engineering and Aerodynamic, Mashhad, Iran

rfemcfd@gmail.com

Abstract: This is a new edition of my papers [1, 2] with a lot of changes. In this paper I'm going to show changes done to the Reduced Finite Element Method (RFEM) that its result will be the most powerful numerical method that has been proposed so far (some forms of this method are so powerful that they can approximate the most complex equations simply Laplace equation!).

Keywords: Reduced finite element method, New computational package, New finite element formulation, New higher order form, New isogeometric analysis.

1 Introduction

The general form of a linear differential equation with boundary conditions can be shown as follows [3, 4]:

$$\begin{aligned} R_{\Omega} &= \mathfrak{L}\,\phi + P \\ R_{\Gamma} &= \mathfrak{M}\,\phi + r \end{aligned} \tag{1}$$

The operators $\mathfrak{L}$ and $\mathfrak{M}$ in equations (1) can be zero-order, odd-order, even-order or a combination of two or all three

$$\begin{aligned} R_{\Omega} &= \mathfrak{L}\,\phi + P = \overset{odd}{\mathfrak{L}}\,\phi + \overset{even}{\mathfrak{L}}\,\phi + \overset{zero}{\mathfrak{L}}\,\phi + P \\ R_{\Gamma} &= \mathfrak{M}\,\phi + r = \overset{odd}{\mathfrak{M}}\,\phi + \overset{even}{\mathfrak{M}}\,\phi + \overset{zero}{\mathfrak{M}}\,\phi + r \end{aligned} \tag{2}$$

The weighted residual relationship for these relationships on any element is

$$\int_{\Omega^e} W_i\, R_{\Omega}\, d\Omega \qquad , \qquad \int_{\Gamma^e} \overline{W_i}\, R_{\Gamma}\, d\Gamma \tag{3}$$

And

$$\int_{\Omega^e} W_i\, R_{\Omega}\, d\Omega + \int_{\Gamma^e} \overline{W_i}\, R_{\Gamma}\, d\Gamma = 0 \tag{4}$$

By inserting (2) in (4) we have

$$\int_{\Omega^e} W_i \overset{odd}{\mathfrak{L}} \phi \, d\Omega + \int_{\Omega^e} W_i \overset{even}{\mathfrak{L}} \phi \, d\Omega + \int_{\Omega^e} W_i \overset{zero}{\mathfrak{L}} \phi \, d\Omega + \int_{\Omega^e} W_i \, P \, d\Omega$$
$$+ \int_{\Gamma^e} \overline{W}_i \overset{odd}{\mathfrak{M}} \phi \, d\Gamma + \int_{\Gamma^e} \overline{W}_i \overset{even}{\mathfrak{M}} \phi \, d\Gamma + \int_{\Gamma^e} \overline{W}_i \overset{zero}{\mathfrak{M}} \phi \, d\Gamma + \int_{\Gamma^e} \overline{W}_i \, r \, d\Gamma = 0 \tag{5}$$

The approximate relation of the field also is equal to

$$\phi \approx \hat{\phi} = \sum_{j=0}^{M} N_j \, \hat{\phi}_j \tag{6}$$

By inserting (6) in (5) we have

$$\int_{\Omega^e} W_i \overset{odd}{\mathfrak{L}} \sum_{j=0}^{M} N_j \, \hat{\phi}_j d\Omega + \int_{\Omega^e} W_i \overset{even}{\mathfrak{L}} \sum_{j=0}^{M} N_j \, \hat{\phi}_j d\Omega + \int_{\Omega^e} W_i \overset{zero}{\mathfrak{L}} \sum_{j=0}^{M} N_j \, \hat{\phi}_j d\Omega + \int_{\Omega^e} W_i \, P \, d\Omega$$
$$+ \int_{\Gamma^e} \overline{W}_i \overset{odd}{\mathfrak{M}} \sum_{j=0}^{M} N_j \, \hat{\phi}_j \, d\Gamma + \int_{\Gamma^e} \overline{W}_i \overset{even}{\mathfrak{M}} \sum_{j=0}^{M} N_j \, \hat{\phi}_j \, d\Gamma + \int_{\Gamma^e} \overline{W}_i \overset{zero}{\mathfrak{M}} \sum_{j=0}^{M} N_j \, \hat{\phi}_j \, d\Gamma \tag{7}$$
$$+ \int_{\Gamma^e} \overline{W}_i \, r \, d\Gamma = 0$$

Together all the elements a comprehensive system of equations is obtained which can be written quite generally as

$$\boldsymbol{K} \, \hat{\boldsymbol{\phi}} = \mathbf{f} \tag{8}$$

And

$$K_{ij} = \sum_{e=1}^{E} K_{ij}^{e} \quad , \quad \mathrm{f}_i = \sum_{e=1}^{E} \mathrm{f}_i^{e} \tag{9}$$

Finally by using of equation (7) can write

$$K_{ij}^{e} = \int_{\Omega^e} W_i \, \mathfrak{L} N_j d\Omega + \int_{\Gamma^e} \overline{W}_i \, \mathfrak{M} \, N_j d\Gamma$$
$$\mathrm{f}_i^{e} = -\int_{\Omega^e} W_i \, P \, d\Omega - \int_{\Gamma^e} \overline{W}_i \, r \, d\Gamma \tag{10}$$
$$\hat{\boldsymbol{\phi}}^{\boldsymbol{T}} = (\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3 \ldots, \hat{\phi}_M)$$

And this is the general form of approximation to a differential equation by the finite element method.

2 New Formulation for Finite Element Method (NFFEM)

Here I will introduce a new formulation for finite element method which its performance is much better than all conventional algorithms of finite element method in CFD. In NFFEM relationship (9) is written as follows:

$$K_{ij}^{*} = \sum_{e=1}^{E} K_{ij}^{e*} \qquad , \qquad \mathrm{f}_i^{*} = \sum_{e=1}^{E} \mathrm{f}_i^{e*} \tag{11}$$

And each of the matrix coefficients are

$$K_{ij}^{e*} = \overset{odd\ e*}{K_{ij}} + \overset{even\ e*}{K_{ij}} + \overset{zero\ e*}{K_{ij}} \tag{12}$$

Where $\overset{odd\ e*}{K_{ij}}$ become

$$\overset{odd\ e*}{K_{ij}} = \sigma_i \int_{\Omega^e} W_i \overset{odd}{\mathfrak{L}} N_j \, d\Omega \tag{13}$$

For $\overset{even\ e*}{K_{ij}}$ we have

$$\overset{even\ e*}{K_{ij}} = \sigma_i \int_{\Omega^e} W_i \overset{even}{\mathfrak{L}} N_j \, d\Omega \tag{14}$$

For $\overset{zero\ e*}{K_{ij}}$ we have

$$\overset{zero\ e*}{K_{ij}} = \sigma_i \int_{\Omega^e} W_i N_j \, d\Omega \tag{15}$$

And for f_i^{e*} we have

$$\mathrm{f}_i^{e*} = \sigma_i \int_{\Omega^e} W_i \, P d\Omega \tag{16}$$

Where σ_i is given by

$$\sigma_i = (1 - \frac{b}{2}(\left|\vec{\beta}_{i^{klm}(x,y,z)}\right| - \vec{\beta}_{i^{klm}(x,y,z)})) \tag{17}$$

These relations are used for boundary integrals the same way, these are relations completely of NFFEM.

In the equations (13) to (16), by choosing $b = 1$ full upwind difference scheme (FUDS), by choosing $b = 0$ central difference scheme (CDS) and by choosing $0 \leq b \leq 1$ hybrid difference scheme (HDS) is obtained. Note that Flux Vector Splitting Methods (FVSM) can also use with NFFEM, in this form $b = 1$ is considered and equations are written separately for positive and negative-terms after that the sum of the equations is done. These relationships are also valid for non-linear operators ($\mathfrak{L}(\phi)$), which leads to $K_{ij}^{e*}(\phi)$. In the equations presented above, $\vec{\beta}_{i^{klm}(x,y,z)}$ is given by

$$\vec{\beta}_{i^{klm}(x,y,z)} = \frac{2(\vec{\beta}_{i^k(x)} + \vec{\beta}_{i^l(y)} + \vec{\beta}_{i^m(z)}) - (\left|\vec{\beta}_{i^k(x)}\right| + \left|\vec{\beta}_{i^l(y)}\right| + \left|\vec{\beta}_{i^m(z)}\right|)}{\left|2(\vec{\beta}_{i^k(x)} + \vec{\beta}_{i^l(y)} + \vec{\beta}_{i^m(z)}) - (\left|\vec{\beta}_{i^k(x)}\right| + \left|\vec{\beta}_{i^l(y)}\right| + \left|\vec{\beta}_{i^m(z)}\right|)\right|} \tag{18}$$

For the central difference scheme σ_i^c can also be used instead of σ_i in equations (13) to (16), where

$$\sigma_i^c = (1 - \frac{1}{2}(\left|\vec{\vec{\beta}}_{i^{klm}(x,y,z)}\right| - \vec{\vec{\beta}}_{i^{klm}(x,y,z)})) \tag{19}$$

Where $\vec{\vec{\beta}}_{i^{klm}(x,y,z)}$ is given by

$$\vec{\vec{\beta}}_{i^{klm}(x,y,z)} = \frac{\left|\vec{\beta}_{i^k(x)} + \vec{\beta}_{i^l(y)} + \vec{\beta}_{i^m(z)}\right| - (\left|\vec{\beta}_{i^k(x)}\right| + \left|\vec{\beta}_{i^l(y)}\right| + \left|\vec{\beta}_{i^m(z)}\right|)}{\left|\left|\vec{\beta}_{i^k(x)} + \vec{\beta}_{i^l(y)} + \vec{\beta}_{i^m(z)}\right| - (\left|\vec{\beta}_{i^k(x)}\right| + \left|\vec{\beta}_{i^l(y)}\right| + \left|\vec{\beta}_{i^m(z)}\right|)\right|} \tag{20}$$

In this form, the number of participating elements in the equation of node i are limited to two elements in 1D, 2D and 3D. The $\vec{\beta}_{i^k(x)}$, $\vec{\beta}_{i^l(y)}$ and $\vec{\beta}_{i^m(z)}$ are equal to

$$\vec{\beta}_{i^k(x)} = \beta_{i^k(x)}\alpha_{i(x)} \quad , \quad \vec{\beta}_{i^l(y)} = \beta_{i^l(y)}\alpha_{i(y)} \quad , \quad \vec{\beta}_{i^m(z)} = \beta_{i^m(z)}\alpha_{i(z)} \tag{21}$$

Where

$$\beta_{i^k(x)} = \left(\frac{x_{i^k} - x_{0^k}}{\left|x_{i^k} - x_{0^k}\right|} + \frac{x_{i^k} - x_{L^k}}{\left|x_{i^k} - x_{L^k}\right|}\right) \quad , \quad \beta_{i^l(y)} = \left(\frac{y_{i^l} - y_{0^l}}{\left|y_{i^l} - y_{0^l}\right|} + \frac{y_{i^l} - y_{L^l}}{\left|y_{i^l} - y_{L^l}\right|}\right)$$
$$\beta_{i^m(z)} = \left(\frac{z_{i^m} - z_{0^m}}{\left|z_{i^m} - z_{0^m}\right|} + \frac{z_{i^m} - z_{L^m}}{\left|z_{i^m} - z_{L^m}\right|}\right) \tag{22}$$

And α_i is the sign of the unknown variable or derivations coefficients (any term that changes the sign of the matrix coefficients), for Euler equations and Navier - Stokes equations in 3D are

$$\alpha_{i(x)} = \frac{A_i}{\left|A_i\right|} \quad , \quad \alpha_{i(y)} = \frac{B_i}{\left|B_i\right|} \quad , \quad \alpha_{i(z)} = \frac{C_i}{\left|C_i\right|} \tag{23}$$

Where A_i, B_i and C_i are the flux Jacobian matrix (derivations coefficients). Or

$$\alpha_{i(x)} = \frac{E_i}{\left|E_i\right|} \quad , \quad \alpha_{i(y)} = \frac{F_i}{\left|F_i\right|} \quad , \quad \alpha_{i(z)} = \frac{G_i}{\left|G_i\right|} \tag{24}$$

Where E_i, F_i and G_i are convective flux vector. Superscript k, l and m represent the terms are located on lines k, l and m (The node i in three dimensions is located on three lines, line k at x direction, line l at y direction, and line m at z direction (see Figures 1 and 2).

For hierarchical shape functions, the shape functions dependent to the sides and element as shown in figures (3) and (4) attributable to the points (these points as $\xi_{0^k}, \eta_{0^l}, \zeta_{0^m}$, $\xi_{i^k}, \eta_{i^l}, \zeta_{i^m}$ and $\xi_{L^k}, \eta_{L^l}, \zeta_{L^m}$ to calculate $\vec{\beta}_i$ also to determine lines k, l and m are used).

When wind direction is constant before and after discontinuity (shock), as an alternative (or as another option for all position), β_i' can be used instead of $\vec{\beta}_i$, where β_i' is

$$\beta'_{i^k{}_{(x)}} = \beta_{i^k{}_{(x)}}\alpha'_{i(x)} \quad , \quad \beta'_{i^l{}_{(y)}} = \beta_{i^l{}_{(y)}}\alpha'_{i(y)} \quad , \quad \beta'_{i^m{}_{(z)}} = \beta_{i^m{}_{(z)}}\alpha'_{i(z)} \tag{25}$$

Where

$$\alpha'_{i(x)} = \frac{\left|\dfrac{\phi_L - \phi_i}{x_{L^k} - x_{i^k}}\right| - \left|\dfrac{\phi_i - \phi_0}{x_{i^k} - x_{0^k}}\right|}{\left\|\dfrac{\phi_L - \phi_i}{x_{L^k} - x_{i^k}}\right| - \left|\dfrac{\phi_i - \phi_0}{x_{i^k} - x_{0^k}}\right\|} \quad , \quad \alpha'_{i(y)} = \frac{\left|\dfrac{\phi_L - \phi_i}{y_{L^l} - y_{i^l}}\right| - \left|\dfrac{\phi_i - \phi_0}{y_{i^l} - y_{0^l}}\right|}{\left\|\dfrac{\phi_L - \phi_i}{y_{L^l} - y_{i^l}}\right| - \left|\dfrac{\phi_i - \phi_0}{y_{i^l} - y_{0^l}}\right\|}$$

$$\alpha'_{i(z)} = \frac{\left|\dfrac{\phi_L - \phi_i}{z_{L^m} - z_{i^m}}\right| - \left|\dfrac{\phi_i - \phi_0}{z_{i^m} - z_{0^m}}\right|}{\left\|\dfrac{\phi_L - \phi_i}{z_{L^m} - z_{i^m}}\right| - \left|\dfrac{\phi_i - \phi_0}{z_{i^m} - z_{0^m}}\right\|} \tag{26}$$

When is used α_i' rather than α_i, always only elements that are located on discontinuity are affected by discontinuity (see Figure 23). Using of α_i' with $b = 1$ gives a low cost scheme compared to other schemes and methods on Direct Numerical Simulation (DNS).

Equation (18) can be written with β_i and $-\beta_i$ instead of the $\vec{\beta}_i$ that its result will be $\beta_{i^{klm}{}_{(x,y,z)}}$, depending on the sign of β_i that we have chosen, the $\beta_{i^{klm}{}_{(x,y,z)}}$ gives the forward or backward differencing scheme.

Note that NFFEM only for quadrilateral and hexahedron elements can be used. Solving the system of equations of the NFFEM can be performed simply by element by element solving method and line by line sweeping method.

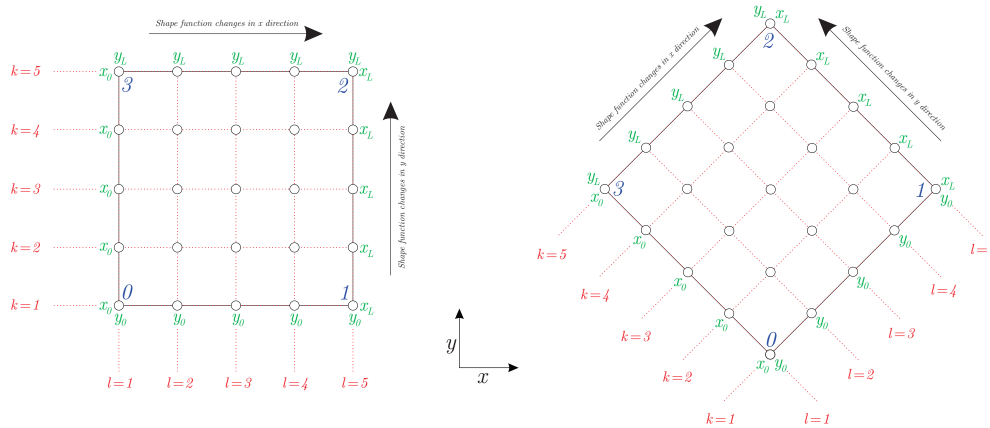

Figure 1: Lines k and l on two fourth-degree elements in two dimensions.

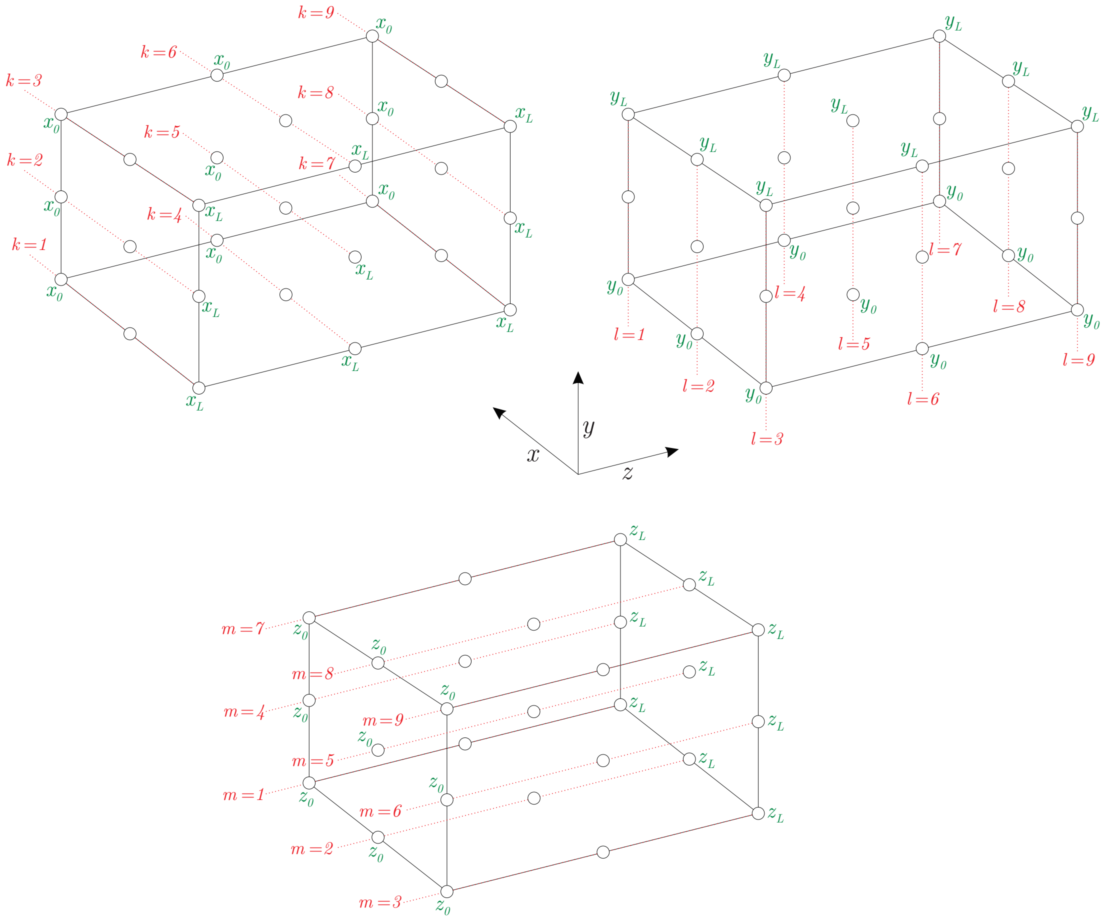

Figure 2: Lines k , l and m on a quadratic element in three dimensions.

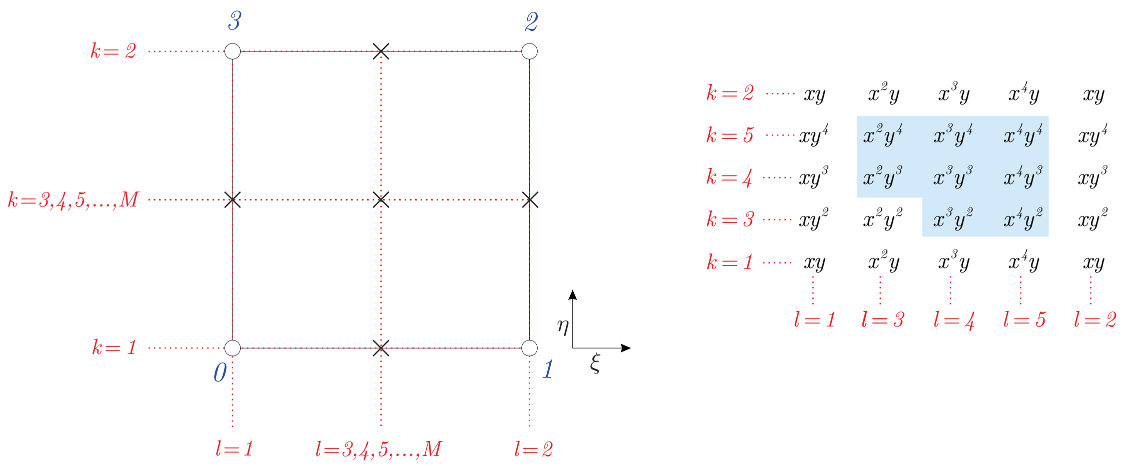

Figure 3: Lines k and l also the places that β is calculated on for a hierarchical element in two dimensions.

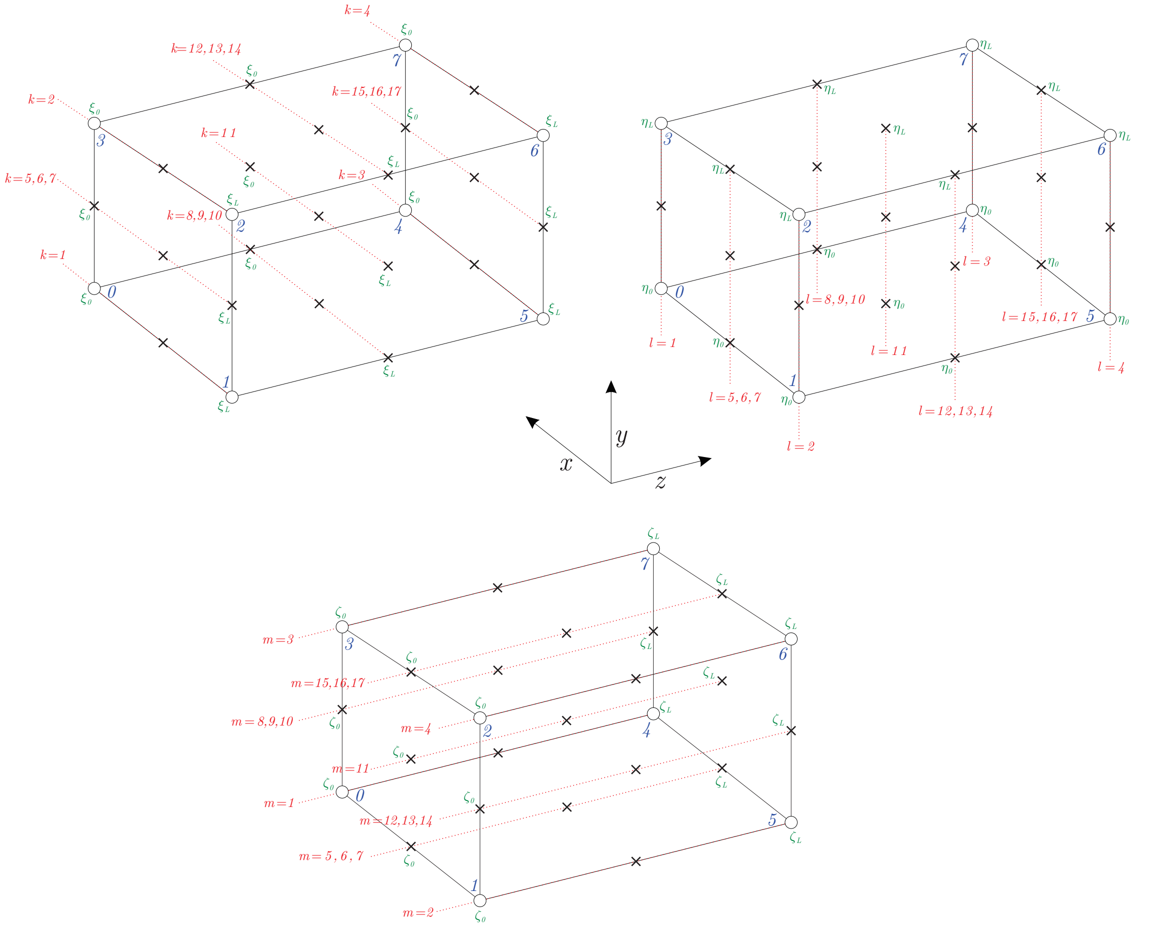

Figure 4: Lines k , l and m also the places that β is calculated on for a fourth-degree hierarchical element in three dimensions.

3 Reduced Finite Element Method (RFEM)

A new method for the finite element formulation presented in the previous section, In this section, I'm going to limit the participating nodes in equation for each node to the nodes that are located on lines k , l and m the node. For this purpose, I introduce reduced elements, characteristics of these elements is as follows:

1- The different elements are used for different directions of operator (see Figures 5 and 6).

2- For each direction of operator, element only in the same direction has DOF (for one direction of shape function is used p - degree function and for other directions is used zero degree function (see Figures 5, 6, 7 and 8).

3- For equation of node i, element only on the lines k , l and m the same node has the node (see Figures 7 and 8).

In RFEM relationship (11) is written as follows:

$$K^{*}_{ij^{k|l|m}} = \sum_{e=1}^{E} K^{e*}_{ij^{k|l|m}} \qquad , \qquad \mathrm{f}^{*}_{i^{klm}} = \sum_{e=1}^{E} \mathrm{f}^{e*}_{i^{klm}} \tag{27}$$

And each of the matrix coefficients are

$$\overset{e*}{K}_{ij^{k|l|m}} = \overset{odd\ e*}{K}_{ij^{k|l|m}} + \overset{even\ e*}{K}_{ij^{k|l|m}} + \overset{zero\ e*}{K}_{ij^{k|l|m}} \tag{28}$$

Where

$$\begin{aligned} \overset{odd\ e*}{K}_{ij^{k|l|m}} &= \overset{odd\ e*}{K}_{ij^{k}(x)}\ or\ \overset{odd\ e*}{K}_{ij^{l}(y)}\ or\ \overset{odd\ e*}{K}_{ij^{m}(z)} \\ \overset{even\ e*}{K}_{ij^{k|l|m}} &= \overset{even\ e*}{K}_{ij^{k}(x)}\ or\ \overset{even\ e*}{K}_{ij^{l}(y)}\ or\ \overset{even\ e*}{K}_{ij^{m}(z)} \\ \overset{zero\ e*}{K}_{ij^{k|l|m}} &= \overset{zero\ e*}{K}_{ij^{k}(x)}\ or\ \overset{zero\ e*}{K}_{ij^{l}(y)}\ or\ \overset{zero\ e*}{K}_{ij^{m}(z)} \end{aligned} \tag{29}$$

For $\overset{odd\ e*}{K}_{ij^{k}(x)}$, $\overset{odd\ e*}{K}_{ij^{l}(y)}$ and $\overset{odd\ e*}{K}_{ij^{m}(z)}$ we have

$$\overset{odd\ e*}{K}_{ij^{k}(x)} = \sigma_i \int_{\Omega^e} W_i \overset{odd}{\mathfrak{L}}_{(x)} N_{j^k} d\Omega \qquad , \qquad \overset{odd\ e*}{K}_{ij^{l}(y)} = \sigma_i \int_{\Omega^e} W_i \overset{odd}{\mathfrak{L}}_{(y)} N_{j^l} d\Omega$$

$$\overset{odd\ e*}{K}_{ij^{m}(z)} = \sigma_i \int_{\Omega^e} W_i \overset{odd}{\mathfrak{L}}_{(z)} N_{j^m} d\Omega \tag{30}$$

For $\overset{even\ e*}{K}_{ij^{k}(x)}$, $\overset{even\ e*}{K}_{ij^{l}(y)}$ and $\overset{even\ e*}{K}_{ij^{m}(z)}$ we have

$$\overset{even\ e*}{K}_{ij^{k}(x)} = \sigma_i \int_{\Omega^e} W_i \overset{even}{\mathfrak{L}}_{(x)} N_{j^k} d\Omega \qquad , \qquad \overset{even\ e*}{K}_{ij^{l}(y)} = \sigma_i \int_{\Omega^e} W_i \overset{even}{\mathfrak{L}}_{(y)} N_{j^l} d\Omega$$

$$\overset{even\ e*}{K}_{ij^{m}(z)} = \sigma_i \int_{\Omega^e} W_i \overset{even}{\mathfrak{L}}_{(z)} N_{j^m} d\Omega \tag{31}$$

For $\overset{zero\ e*}{K}_{ij^{k}(x)}$, $\overset{zero\ e*}{K}_{ij^{l}(y)}$ and $\overset{zero\ e*}{K}_{ij^{m}(z)}$ we have

$$\overset{zero\ e*}{K}_{ij^{k}(x)} = \frac{\sigma_i}{D} \int_{\Omega^e} W_i N_{j^k} d\Omega \qquad , \qquad \overset{zero\ e*}{K}_{ij^{l}(y)} = \frac{\sigma_i}{D} \int_{\Omega^e} W_i N_{j^l} d\Omega$$

$$\overset{zero\ e*}{K}_{ij^{m}(z)} = \frac{\sigma_i}{D} \int_{\Omega^e} W_i N_{j^m} d\Omega \tag{32}$$

Where D is dimensions. Note that for the $\overset{zero}{\mathfrak{L}}$ operator, unknown variable is written in all directions and each direction is no belongs to particular unknown variable. And for $\overset{e*}{\mathrm{f}}_{i^{klm}}$ we have

$$\overset{e*}{\mathrm{f}}_{i^{klm}} = \sigma_i P_i \int_{\Omega^e} W_i \, d\Omega \tag{33}$$

For approximating the mixed derivatives we will need to more DOF for example; if in three dimensions, the mixed derivative be in two-direction, like following derivative

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} = \overset{mix}{\mathfrak{L}}_{xy}\, \phi \tag{34}$$

We use the shape function with two DOF in x and y directions (see Figure 9). This elements are used only for mixed operator.

When we use the hierarchical shape functions, for some equations have two unknown variable (ϕ_i and a_i, see Figure 6), the additional equation for additional unknown can be written as follows:

$$3\phi_i - \left[\sum_{j=0}^{L} N_{j^l}\Big|_i \phi_j + \sum_{j=0}^{L} N_{j^l}\Big|_i \mathrm{a}_j\right] + \left[\sum_{j=0}^{L} N_{j^m}\Big|_i \phi_j + \sum_{j=0}^{L} N_{j^m}\Big|_i \mathrm{a}_j\right]$$
$$+ \left[\sum_{j=0}^{L} N_{j^k}\Big|_i \phi_j + \sum_{j=0}^{L} N_{j^k}\Big|_i \mathrm{a}_j\right] = 0 \qquad (35)$$

RFEM can be used to isogeometric analysis as well (see the following sections and examples).

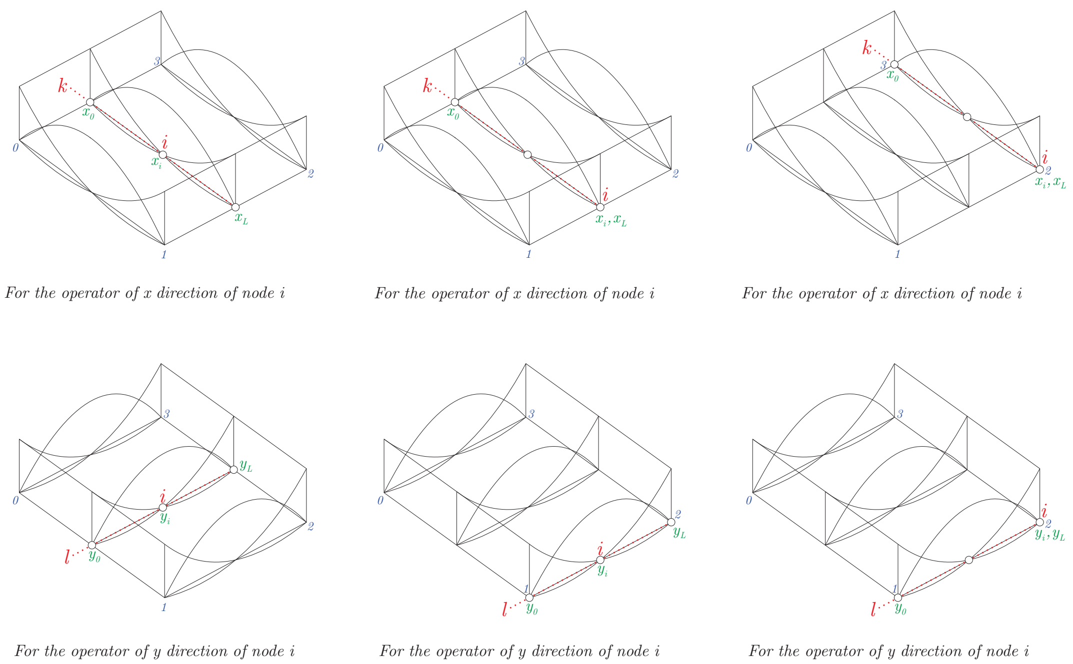

Figure 5: A reduced standard quadratic element in two-dimensional, lines k and l also the participating nodes in the equation of node i for each of them.

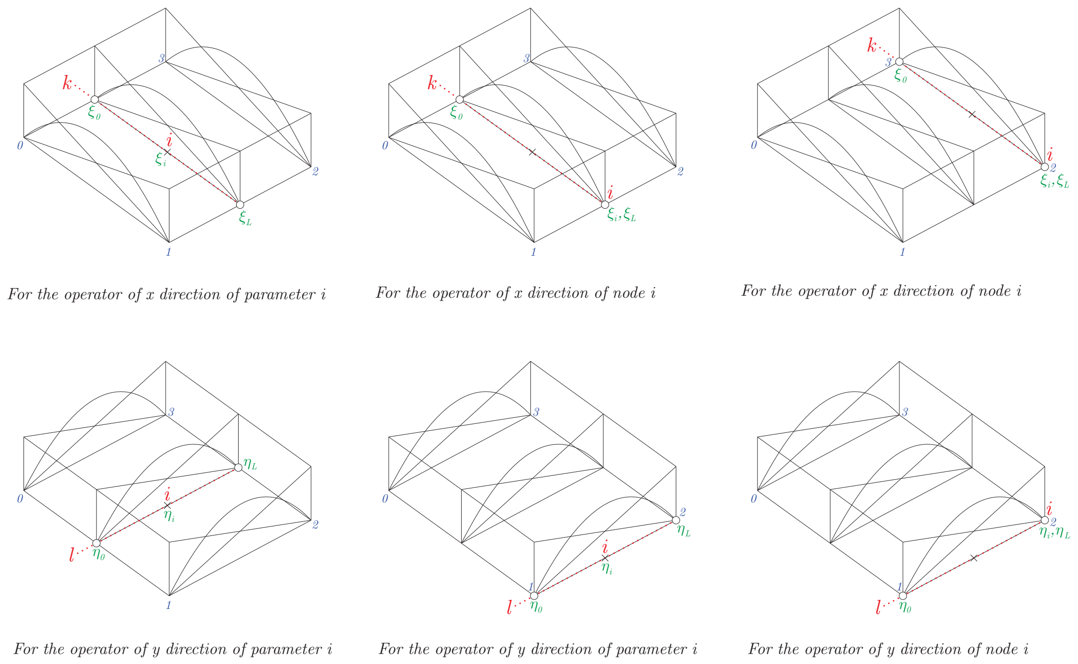

Figure 6: A reduced hierarchical quadratic element in two-dimensional, lines k and l also the participating nodes in the equation of node i for each of them.

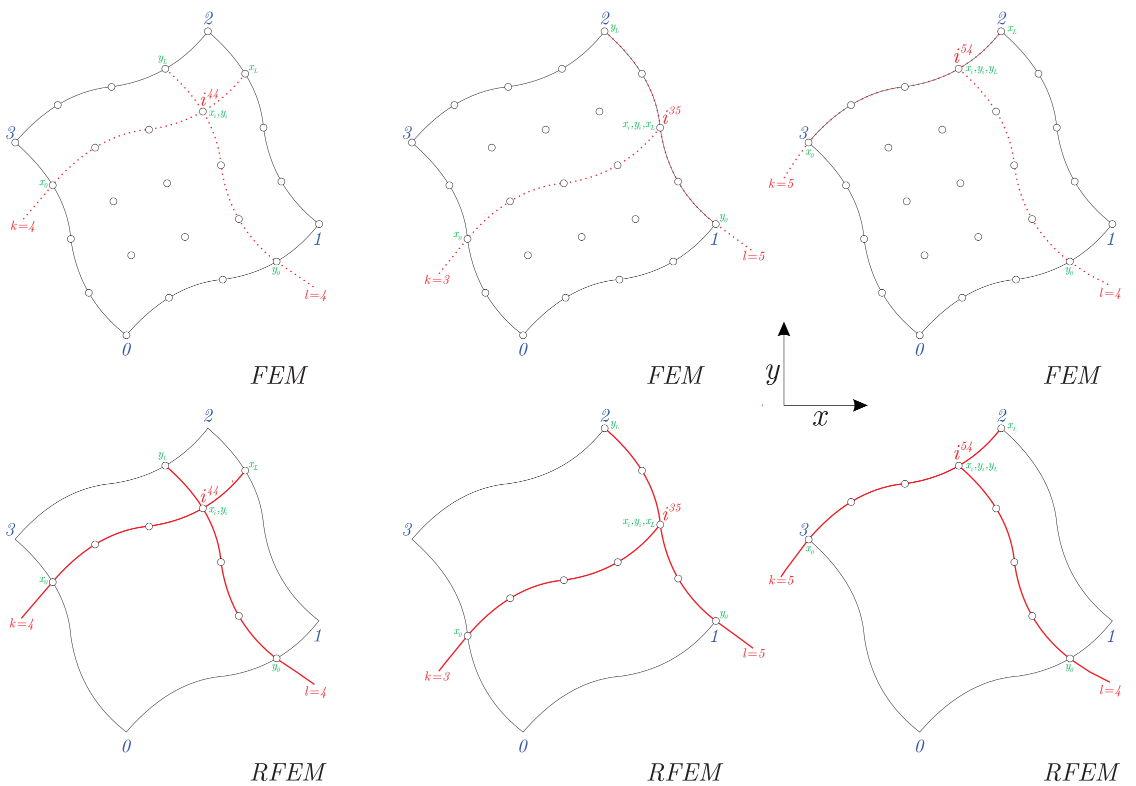

Figure 7: Nodes involved in the equation of node i for FEM and RFEM on several fourth-degree elements in two dimensions.

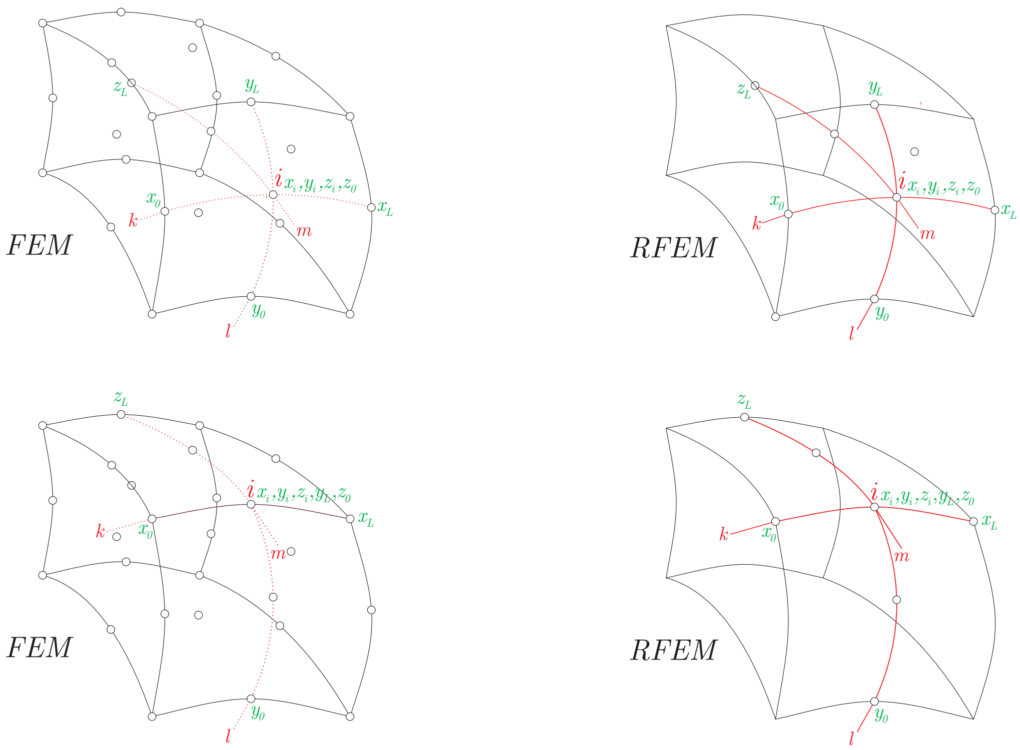

Figure 8: Nodes involved in the equation of node i for FEM and RFEM on two quadratic elements in three dimensions.

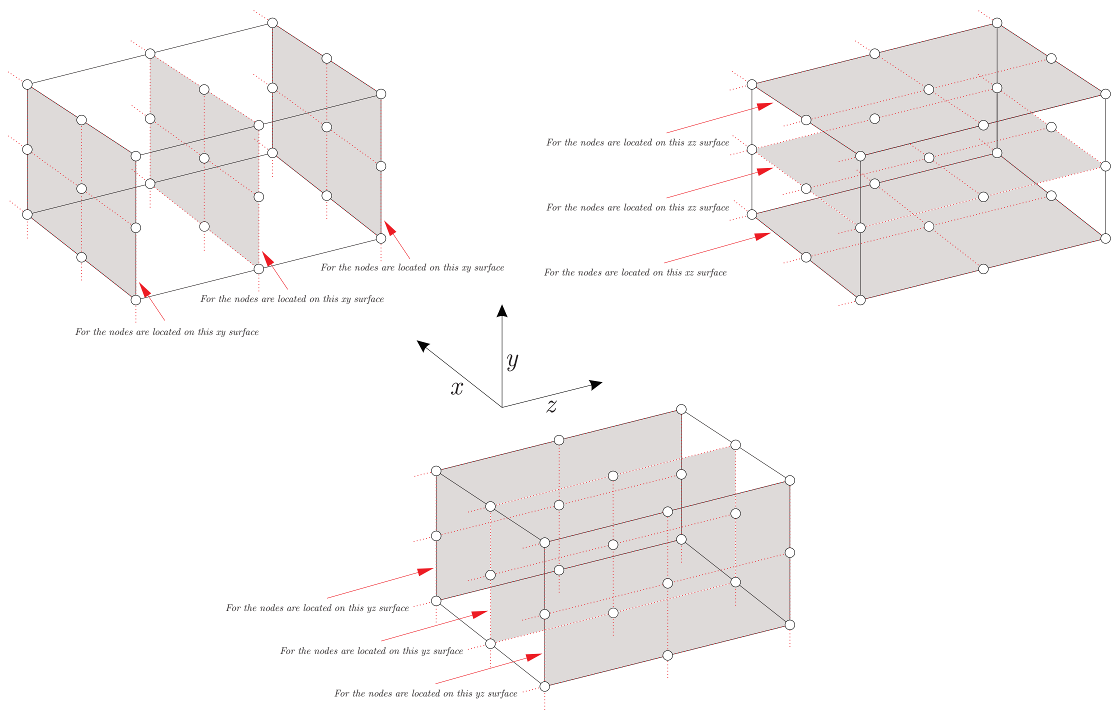

Figure 9: A reduced quadratic element in three-dimensional with two DOF for approximation the mixed derivatives $\partial^2\phi/\partial x\partial y$, $\partial^2\phi/\partial x\partial z$ and $\partial^2\phi/\partial y\partial z$.

4 A new technique to approximate complete second order differential equation

Complete second order differential equation can be written as follows:

$$D\frac{d^2\phi}{dx^2} + A\frac{d\phi}{dx} + J\phi = \mathrm{f} \tag{36}$$

The problem in approximation of equation (36) is related to the first-order derivative when $|A|$ is a big number, by derivatives coefficients of equation (36) can write

$$|Pe| = \frac{|A|\,\Delta x}{D} \tag{37}$$

Where $|Pe|$ is absolute of Peclet number. Now, I will introduce a technique to approximate equation (36) by FEM and RFEM on liner shape functions based on the Peclet number, in this case b is written as follows:

$$b = \theta_{i(x,y,z)} \tag{38}$$

Equation (38) give a Hybrid Difference Scheme (HDS), where $\theta_{i(x,y,z)}$ is "Upwind Parameter" and is chosen in the range $0 \le \theta_{i(x,y,z)} \le 1$, it is clear that there are many choices that can be used for. I used the following equation based on the Peclet number for it

$$\theta^{Pe}_{i(x,y,z)} = 1 - \frac{1}{2\left|Pe_{i^{klm}(x,y,z)}\right|} \tag{39}$$

Where $\left|Pe_{i^{klm}(x,y,z)}\right|$ is given by

$$\left|Pe_{i^{klm}(x,y,z)}\right| = \frac{\left|Pe_{i^{k}(x)}\right|^2 + \left|Pe_{i^{l}(y)}\right|^2 + \left|Pe_{i^{m}(z)}\right|^2}{\left|Pe_{i^{k}(x)}\right| + \left|Pe_{i^{l}(y)}\right| + \left|Pe_{i^{m}(z)}\right|} \tag{40}$$

And

$$\left|Pe_{i^{k}(x)}\right| = \frac{|A_i|\,\Delta x}{D_i} \quad , \quad \left|Pe_{i^{l}(y)}\right| = \frac{|B_i|\,\Delta y}{D_i} \quad , \quad \left|Pe_{i^{m}(z)}\right| = \frac{|C_i|\,\Delta z}{D_i} \tag{41}$$

Where

$$\Delta x = x_{L^k} - x_{0^k} \quad , \quad \Delta y = y_{L^l} - y_{0^l} \quad , \quad \Delta z = z_{L^m} - z_{0^m} \tag{42}$$

The A_i, B_i and C_i are the flux Jacobian matrix (derivations coefficients). I write $\theta_{i(x,y,z)}$ based on the Peclet number, when the Peclet number is very high (or as another option for all Peclet numbers), $\theta_{i(x,y,z)}$ can be written based on the gradient of unknown variable, I've used the following equation for θ_i as a better option than [1]

$$\theta^{Gr}_{i(x,y,z)} = 1 - \frac{1}{\left|r_{i^{klm}(x,y,z)}\right|^{0.5}} \tag{43}$$

Equation (43) is a basic choice to $\theta^{Gr}_{i(x,y,z)}$, It is clear that there are many choices that can be used for, and $\left|r_{i^{klm}(x,y,z)}\right|$ in equation (43) is given by

$$\left|r_{i^{klm}(x,y,z)}\right| = max\left[\left|r_{i^{k}(x)}\right|, \left|r_{i^{l}(y)}\right|, \left|r_{i^{m}(z)}\right|\right] \tag{44}$$

A much better form to calculate r_i compared to [1] can be written as follows:

$$\left|r_{i^{k}(x)}\right| = \overset{a}{g}_{i^{k}(x)}\left[\frac{max\left[\left|\frac{\phi_L - \phi_i}{x_{L^k} - x_{i^k}}\right|, \left|\frac{\phi_i - \phi_0}{x_{i^k} - x_{0^k}}\right|\right]}{min\left[\left|\frac{\phi_L - \phi_i}{x_{L^k} - x_{i^k}}\right|, \left|\frac{\phi_i - \phi_0}{x_{i^k} - x_{0^k}}\right|\right]}\right] + \overset{b}{g}_{i^{k}(x)}\left[min\left[\frac{\left|\frac{\overset{e}{\phi}_L - \overset{e}{\phi}_0}{\overset{e}{x}_{L^k} - \overset{e}{x}_{0^k}}\right|}{\left|\frac{\overset{e-1}{\phi}_L - \overset{e-1}{\phi}_0}{\overset{e-1}{x}_{L^k} - \overset{e-1}{x}_{0^k}}\right|}, \frac{\left|\frac{\overset{e}{\phi}_L - \overset{e}{\phi}_0}{\overset{e}{x}_{L^k} - \overset{e}{x}_{0^k}}\right|}{\left|\frac{\overset{e+1}{\phi}_L - \overset{e+1}{\phi}_0}{\overset{e+1}{x}_{L^k} - \overset{e+1}{x}_{0^k}}\right|}\right]\right]$$

$$\left|r_{i^{l}(y)}\right| = \overset{a}{g}_{i^{l}(y)}\left[\frac{max\left[\left|\frac{\phi_L - \phi_i}{y_{L^l} - y_{i^l}}\right|, \left|\frac{\phi_i - \phi_0}{y_{i^l} - y_{0^l}}\right|\right]}{min\left[\left|\frac{\phi_L - \phi_i}{y_{L^l} - y_{i^l}}\right|, \left|\frac{\phi_i - \phi_0}{y_{i^l} - y_{0^l}}\right|\right]}\right] + \overset{b}{g}_{i^{l}(y)}\left[min\left[\frac{\left|\frac{\overset{e}{\phi}_L - \overset{e}{\phi}_0}{\overset{e}{y}_{L^l} - \overset{e}{y}_{0^l}}\right|}{\left|\frac{\overset{e-1}{\phi}_L - \overset{e-1}{\phi}_0}{\overset{e-1}{y}_{L^l} - \overset{e-1}{y}_{0^l}}\right|}, \frac{\left|\frac{\overset{e}{\phi}_L - \overset{e}{\phi}_0}{\overset{e}{y}_{L^l} - \overset{e}{y}_{0^l}}\right|}{\left|\frac{\overset{e+1}{\phi}_L - \overset{e+1}{\phi}_0}{\overset{e+1}{y}_{L^l} - \overset{e+1}{y}_{0^l}}\right|}\right]\right] \tag{45}$$

$$\left|r_{i^{m}(z)}\right| = \overset{a}{g}_{i^{m}(z)}\left[\frac{max\left[\left|\frac{\phi_L - \phi_i}{z_{L^m} - z_{i^{lm}}}\right|, \left|\frac{\phi_i - \phi_0}{z_{i^m} - z_{0^m}}\right|\right]}{min\left[\left|\frac{\phi_L - \phi_i}{z_{L^m} - z_{i^{lm}}}\right|, \left|\frac{\phi_i - \phi_0}{z_{i^m} - z_{0^m}}\right|\right]}\right] + \overset{b}{g}_{i^{m}(z)}\left[min\left[\frac{\left|\frac{\overset{e}{\phi}_L - \overset{e}{\phi}_0}{\overset{e}{z}_{L^m} - \overset{e}{z}_{0^m}}\right|}{\left|\frac{\overset{e-1}{\phi}_L - \overset{e-1}{\phi}_0}{\overset{e-1}{z}_{L^m} - \overset{e-1}{z}_{0^m}}\right|}, \frac{\left|\frac{\overset{e}{\phi}_L - \overset{e}{\phi}_0}{\overset{e}{z}_{L^m} - \overset{e}{z}_{0^m}}\right|}{\left|\frac{\overset{e+1}{\phi}_L - \overset{e+1}{\phi}_0}{\overset{e+1}{z}_{L^m} - \overset{e+1}{z}_{0^m}}\right|}\right]\right]$$

Where

$$\overset{a}{g}_{i^{k}(x)} = \left[(1-c)\left|\beta_{i^{k}(x)}\right|\right] \quad , \quad \overset{a}{g}_{i^{l}(y)} = \left[(1-c)\left|\beta_{i^{l}(y)}\right|\right] \quad , \quad \overset{a}{g}_{i^{m}(z)} = \left[(1-c)\left|\beta_{i^{m}(z)}\right|\right] \tag{47}$$

And

$$\overset{b}{g}_{i^{k}(x)} = \left[c\left|\beta_{i^{k}(x)}\right| + (1-\left|\beta_{i^{k}(x)}\right|)\right] \quad , \quad \overset{b}{g}_{i^{l}(y)} = \left[c\left|\beta_{i^{l}(y)}\right| + (1-\left|\beta_{i^{l}(y)}\right|)\right]$$

$$\overset{b}{g}_{i^{m}(z)} = \left[c\left|\beta_{i^{m}(z)}\right| + (1-\left|\beta_{i^{m}(z)}\right|)\right] \tag{46}$$

Where c for FUDS is 1 and for CDS and HDS is 0. The solution can starts with $\theta^{Gr}_{i(x,y,z)} = 1$. Note that for liner shape functions only first term of equation (45) is used and second term will be zero.

5 A new shape functions (limited shape functions) for general problems

However, the hybrid method is provided above has very good performance, but when the Peclet number is infinite, its accuracy is of the first order, also the equation (38) can be used only for the linear shape functions, in this section another technique is provided for eliminating oscillations that can be used for higher-order shape functions in any degree. This technique works directly on the shape functions (this technique is equivalent to creating new shape functions) and is applicable for central difference scheme (CDS), hybrid difference scheme (HDS) and full upwind difference scheme (FUDS), for this purpose, the shape functions are written as following

$$\begin{aligned} N_0^{EDL,FVL} &= (1-\delta_i)N_0^{(p)} + \delta_i N_0^{(ref)} \\ N_j^{EDL,FVL} &= (1-\delta_i)N_j^{(p)} + \delta_i N_j^{(ref)} \\ N_L^{EDL,FVL} &= (1-\delta_i)N_L^{(p)} + \delta_i N_L^{(ref)} \end{aligned} \tag{48}$$

The equation (46) is non-oscillatory shape function, where δ_i is Element Degree Limiter, Field Variable Limiter (EDL, FVL), and $N^{(p)}$ can be any function of $p \geq 2$ (for example, the functions of Lagrangian, Serendipity and Bezier for standard shape functions and the functions of Legendre, Chebyshev, Fourier, etc. for hierarchical shape functions). $N_0^{(ref)}$, $N_j^{(ref)}$ and $N_L^{(ref)}$ in equation (48) are reference functions, original function for them become

$$N_0^{(ref)} = N_0^{(1)} \quad , \quad N_j^{(ref)} = 0 \quad , \quad N_L^{(ref)} = N_L^{(1)} \tag{49}$$

Where $N_0^{(1)}$ and $N_L^{(1)}$ are liner shape functions. The following functions can also be used as reference function on the Lagrangian, Serendipity and Bezier functions to make the shape functions of non-oscillatory

$$N_0^{(ref)} = eN_L \ \ , \ \ N_j^{(ref)} = S_j^0 N_0 + S_j^L N_L + N_j \ \ , \ \ N_L^{(ref)} = eN_0 \tag{50}$$

And for boundary elements become

$$\begin{aligned} &N_0^{(ref)} = N_0 + eN_L \quad , \quad N_j^{(ref)} = S_j^L N_L + N_j \quad , \quad N_L^{(ref)} = 0 \\ &N_0^{(ref)} = 0 \quad , \quad N_j^{(ref)} = S_j^0 N_0 + N_j \quad , \quad N_L^{(ref)} = N_L + eN_0 \end{aligned} \tag{51}$$

Where e is a constant coefficient and N_0, N_j and N_L can be the Lagrangian, Serendipity or Bezier functions and degree them can be chosen *1* or *p* (for *1*-degree and *p*-degree the equations and results are different). S_j^0 and S_j^L are a constant coefficients and are given by

$$S_j^0 = (1-e)\frac{\left(\dfrac{\xi_j-\xi_0}{\xi_L-\xi_0}\right)^{\omega}}{\sum\limits_{j=1}^{p-1}\left(\dfrac{\xi_j-\xi_0}{\xi_L-\xi_0}\right)^{\omega}} \quad , \quad S_j^L = (1-e)\frac{\left(\dfrac{\xi_L-\xi_j}{\xi_L-\xi_0}\right)^{\omega}}{\sum\limits_{j=1}^{p-1}\left(\dfrac{\xi_L-\xi_j}{\xi_L-\xi_0}\right)^{\omega}} \tag{52}$$

Where ξ_j is the location of the nodes (control points) and ω is the weight coefficient (optional). For $p \geq 3$ the hierarchical shape functions with $N_j^{(ref)} = 0$ can be used. e is chosen as, $e = 1 - p$ for $\omega = \infty$, $e = -1$ for $\omega = 0$ and $e = 1/1 - p$ for $\omega = -\infty$. For other ω the value of e is different (for example, for $p=3$, when $w=1$, $e=-1.25$ and when $w=2$, $e=-1.5$ or when $w=-1$, $e=-0.8$ and when $w=-2$, $e=-2/3$). Another reference function is

$$\begin{aligned} N_0^{(ref)} &= \frac{1}{2}(1+\alpha_i)(N_0+eN_L) \\ N_j^{(ref)} &= \frac{1}{2}\left[(1+\alpha_i)S_j^L N_L+(1-\alpha_i)S_j^0 N_0\right]+N_j \\ N_L^{(ref)} &= \frac{1}{2}(1-\alpha_i)(N_L+eN_0) \end{aligned} \tag{53}$$

When $\alpha_i=1$ equation (53) give backward difference approximation and when $\alpha_i=-1$ give forward difference approximation. The δ_i in equation (48) has two range:

1- Using the δ_i as EDL by choosing it in the range $0<\delta_i^{EDL}\le\infty$.
2- Using the δ_i as FVL by choosing it in the range $-\infty\le\delta_i^{FVL}<0$.

Note: if the reference function is greater than the shape function the sign of EDL and FVL will change. The EDL can be used only in the directions of the shape function (directions of operator) that $\beta_i\neq 0$ is for, and the FVL is used only in the directions of the shape function that $\beta_i=0$ is for or inverse, or the FVL and EDL can be added on all nodes with identical or non-identical values, and their relationship is written as follows:

$$\delta_i=\left[|\beta_i|\delta_i^{(EDL,FVL)1}+(1-|\beta_i|)\delta_i^{(EDL,FVL)2}\right]\theta_{i(x,y,z)} \tag{54}$$

Where $\theta_{i(x,y,z)}$ is given by equation (39) or (43). A constant value of $\theta_{i(x,y,z)}$ ($\theta_{i(x,y,z)}=1$) can also be used as low cost form, note that when θ_i is written based on Peclet number, when the Peclet number is infinite the $\theta_{i(x,y,z)}$ will be 1 (this paper is based on $\theta_{i(x,y,z)}=1$ and I show when $\theta_{i(x,y,z)}=1$ is considered, very accurate solutions can be achieved for all conditions). The δ_i is applied to shape functions as one-dimensional (separately for each direction of shape functions). The value of the $\delta_i^{(EDL,FVL)1}$ and $\delta_i^{(EDL,FVL)2}$ depending on the application is, forms that can make by using of EDL, FVL is very high, here I am presenting only some of them which are included;

Full Upwind Difference Scheme ($b=1$)

Form 1, for $p=2$;

- To use equation (53) as reference function with $p=1$, $e=-1$ and $\omega=0$ on equation (52).
- To choose in equation (54)
  A- $\delta_i^{(EDL,FVL)1}=-1$, $\delta_i^{(EDL,FVL)2}=1$.
  B- $\delta_i^{(EDL,FVL)1}=-100$, $\delta_i^{(EDL,FVL)2}=100$.
  C- $\delta_i^{(EDL,FVL)1}=-1000$, $\delta_i^{(EDL,FVL)2}=100$.

Form 2, for $p\ge 3$;

- To use equation (49) as reference function.
- To choose in equation (54)
  A- $\delta_i^{(EDL,FVL)1}=\delta_i^{(EDL,FVL)2}=-100$.
  B- $\delta_i^{(EDL,FVL)1}=\delta_i^{(EDL,FVL)2}=-1$ (this is a very inexpensive scheme with high performance and always is used with $\theta_{i(x,y,z)}=1$ on equation (54)).

Hybrid Difference Scheme ($b = \theta_{i(x,y,z)}$)

Hybrid Difference Scheme (HDS) is achieved by placing $b = \theta_{i(x,y,z)}$ on the forms of the FUDS.

Central Difference Scheme ($b = 0$)

Form 3, for $p = 2$;

- To use equation (53) as reference function with $p = 1$, $e = -1$ and $\omega = 0$ on equation (52).
- To choose $\delta_i^{(EDL,FVL)1} = 1000\beta_i$, $\delta_i^{(EDL,FVL)2} = 1000$ in equation (54).

Form 4, for $p = 3$;

- To use equation (53) as reference function with $p = 1$, $e = -1$ and $\omega = 0$ on equation (52).
- To write $\delta_i^{(EDL,FVL)1} = \frac{-ap}{2}(1+\vec{\beta}_i) + \frac{ap+p+1}{2p}(1-\vec{\beta}_i)$ and $\delta_i^{(EDL,FVL)2} = -ap$ in equation (54).
- To choose $a = 30$ and $p = 3$ in this form.

Form 5, for $p = 3$;

- To use equation (50) as reference function with $p = 3$, $e = -1.25$ and $\omega = 1$ on equation (52).
- To choose $\delta_i^{(EDL,FVL)1} = \frac{-p}{2}(1+\vec{\beta}_i) + \frac{-p-1}{2}(1-\vec{\beta}_i)$, $\delta_i^{(EDL,FVL)2} = -p-1$ in equation (54).

Form 6, for $p = 3$;

- To use equation (49) as reference function.
- To write $\delta_i^{(EDL,FVL)1} = \frac{-ap}{2}(1+\vec{\beta}_i) + \frac{ap+p+1}{2p}(1-\vec{\beta}_i)$ and $\delta_i^{(EDL,FVL)2} = -ap$ in equation (54).
- A- To choose $a = 30$ and $p = 3$ in this form.
- B- To choose $a = 1/3$ and $p = 3$ in this form (this form is like form 2-B but on CDS).

As other form for central difference scheme (CDS) in equation (48), N_j is written twice, once with α_0 and once with α_L

$$\begin{aligned}
N_0^{EDL,FVL} &= (1-\delta_i)N_0^{(p)} + \delta_i N_0^{(ref)} \\
N_j^{EDL,FVL} &= \frac{1}{2}(|\alpha_0| + \alpha_0)(1-\delta_i)N_j^{(p)} + \frac{1}{2}(|\alpha_0| + \alpha_0)\delta_i N_j^{(ref)} \\
N_j^{EDL,FVL} &= \frac{1}{2}(|\alpha_L| - \alpha_L)(1-\delta_i)N_j^{(p)} + \frac{1}{2}(|\alpha_L| - \alpha_L)\delta_i N_j^{(ref)} \\
N_L^{EDL,FVL} &= (1-\delta_i)N_L^{(p)} + \delta_i N_L^{(ref)}
\end{aligned} \tag{55}$$

In all regions $\alpha_0 = \alpha_L$ or one of them is zero as a result, one of the equations is removed but in discontinuous regions $\alpha_0 \neq \alpha_L$ and will have two equations. This form is always written with $\delta_i^{(EDL,FVL)1} = 0$ and the scheme is in all regions (including regions of large gradients such as shock waves) quite central difference. This form give the best result for all positions and always is used with $\theta_{i(x,y,z)} = 1$ on equation (54).

Form 7, for $p = 2$;

- To use equation (53) as reference function with $p = 1$, $e = -1$ and $\omega = 0$ on equation (52).
- To choose $\delta_i^{(EDL,FVL)1} = 0$, $\delta_i^{(EDL,FVL)2} = 2$ in equation (54).

Form 8, for $p = 3$;

- To use equation (49) as reference function.
- To write $\delta_i^{(EDL,FVL)1} = 0$ and $\delta_i^{(EDL,FVL)2} = \dfrac{-a}{2}(1 - \vec{\beta}_i^{g}) + \dfrac{p+1}{2}(1 + \vec{\beta}_i^{g})$ in equation (54).
- A- To choose $a = 2$ and $p = 3$ in this form.
- B- To choose $a = 0$ and $p = 3$ in this form.

Where $\vec{\beta}_i^{g}$ is given by

$$\vec{\beta}^{g}_{i^k(x)} = \beta^{g}_{i^k(x)}\,\alpha_{i(x)} \quad , \quad \vec{\beta}^{g}_{i^l(y)} = \beta^{g}_{i^l(y)}\,\alpha_{i(y)} \quad , \quad \vec{\beta}^{g}_{i^m(z)} = \beta^{g}_{i^m(z)}\,\alpha_{i(z)} \tag{56}$$

Where

$$\beta^{g}_{i^k(x)} = \left(\frac{(x_{i^k} - x_{0^k}) - (x_{i^k} - x_{L^k})}{\left|(x_{i^k} - x_{0^k}) - (x_{i^k} - x_{L^k})\right|} \right) \quad , \quad \beta^{g}_{i^l(y)} = \left(\frac{(y_{i^l} - y_{0^l}) - (y_{i^l} - y_{L^l})}{\left|(y_{i^l} - y_{0^l}) - (y_{i^l} - y_{L^l})\right|} \right)$$
$$\beta^{g}_{i^m(z)} = \left(\frac{(z_{i^m} - z_{0^m}) - (z_{i^m} - z_{L^m})}{\left|(z_{i^m} - z_{0^m}) - (z_{i^m} - z_{L^m})\right|} \right) \tag{57}$$

These plans are available for the Lagrangian shape functions but for $p \geq 3$ can be used on hierarchical shape functions without change. The schemes presented above can be written for Bezier functions as well.

6 New isogeometric analysis and new functions

In this section, I'm going by reference functions presented in the previous section, I introduce a new isogeometric analysis. The isogeometric analysis [5] is done by the B-Spline and NURBS functions, although the reference functions presented above are applicable with B-Spline and NURBS functions, here I introduce a new Isogeometric Analysis Functions that are closer to the standard finite element method and more comfortable for the use in CFD and engineering, I made the following algorithm to make these functions on the Bezier functions

$$\begin{aligned} &N_0^{(IGA)} = \frac{1}{2} N_0^{(Bezier)} \quad , \quad N_1^{(IGA)} = \frac{1}{2} N_0^{(Bezier)} + N_1^{(Bezier)} \\ &N_{2,3,4,\ldots,L-2}^{(IGA)} = N_{2,3,4,\ldots,L-2}^{(Bezier)} \\ &N_{L-1}^{(IGA)} = \frac{1}{2} N_L^{(Bezier)} + N_{L-1}^{(Bezier)} \quad , \quad N_L^{(IGA)} = \frac{1}{2} N_L^{(Bezier)} \end{aligned} \tag{58}$$

Note that $N_{2,3,4,\ldots,L-2}^{(IGA)}$ are local and can add as hierarchical as well, for $p = 2$ these functions are equivalent to the B-Spline functions and for $p = 3$ are similar the PHT-spline functions work of [6] and for more than a third degree are completely new. The equation (58) for first and last boundary elements become

$$
\begin{aligned}
&N_0^{(IGA)} = N_0^{(Bezier)} \quad , \quad N_1^{(IGA)} = N_1^{(Bezier)} \\
&N_{2,3,4,\dots,L-2}^{(IGA)} = N_{2,3,4,\dots,L-2}^{(Bezier)} \\
&N_{L-1}^{(IGA)} = \frac{1}{2} N_L^{(Bezier)} + N_{L-1}^{(Bezier)} \quad , \quad N_L^{(IGA)} = \frac{1}{2} N_L^{(Bezier)}
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
&N_0^{(IGA)} = \frac{1}{2} N_0^{(Bezier)} \quad , \quad N_1^{(IGA)} = \frac{1}{2} N_0^{(Bezier)} + N_1^{(Bezier)} \\
&N_{2,3,4,\dots,L-2}^{(IGA)} = N_{2,3,4,\dots,L-2}^{(Bezier)} \\
&N_{L-1}^{(IGA)} = N_{L-1}^{(Bezier)} \quad , \quad N_L^{(IGA)} = N_L^{(Bezier)}
\end{aligned} \tag{60}
$$

For IGA functions, the liner functions to use in equations (49), (50) and (53) are a liner functions between two control points (see Figure 10), while the *p*-order functions for using in equations (50) and (53) are the same IGA functions. For functions (60) can be write

Form 9, for $p = 2$;

- To use equation (53) as reference function with $S_j^0 = S_j^L = 2$ ($e = -1$ and $\omega = 0$ on equation (52)).
- To choose $\delta_i^{(EDL,FVL)1} = 0$, $\delta_i^{(EDL,FVL)2} = 1$ in equation (54).

Form 10, for $p = 3$ on non-boundary elements;

- To use equation (49) as reference function.
- To choose $\delta_i^{(EDL,FVL)1} = 0$, $\delta_i^{(EDL,FVL)2} = \frac{-7}{4}(1 - \vec{\beta}_i^g) + \frac{7}{4}(1 + \vec{\beta}_i^g)$ in equation (54).

On boundary elements;

- To choose $\delta_i^{(EDL,FVL)1} = 0$, $\delta_i^{(EDL,FVL)2} = \frac{-3}{2}(1 - \vec{\beta}_i^g) + \frac{1143}{1000}(1 + \vec{\beta}_i^g)$ in equation (54).

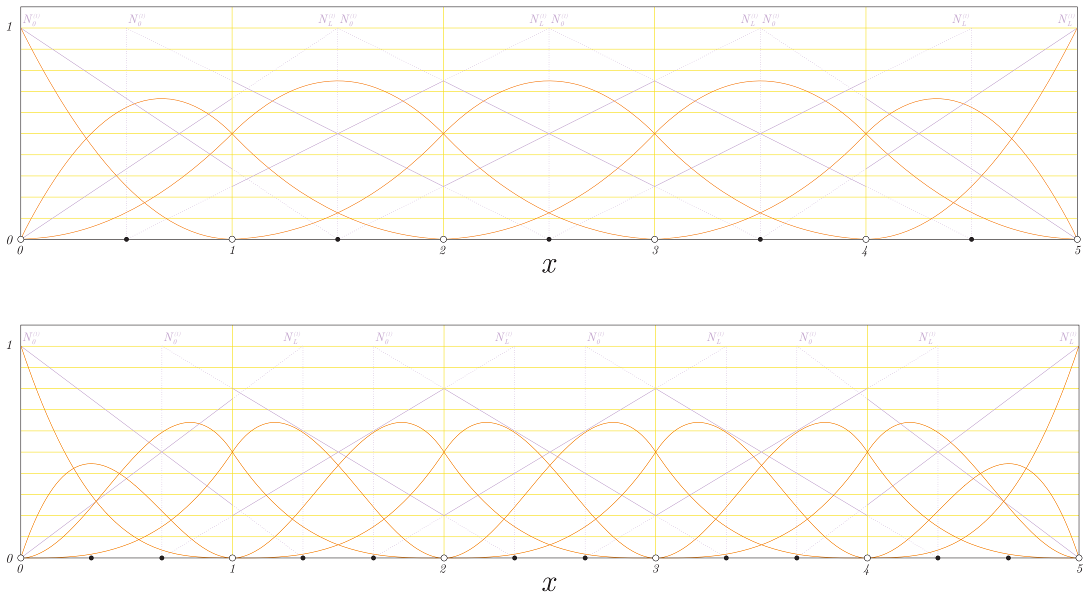

Figure 10: Liner functions for quadratic and cubic IGA functions.

7 Solutions and Examples

In this section, some examples that have been solved by standard shape functions that are including functions of the Lagrangian and hierarchical shape functions that are including functions of the Legendre, Chebyshev (first and second kind) and Fourier Sine Series are presented. It explained that the volume examples were solved to test RFEM is very high and here are just a few of them select and presented. Examples include;

1- 1D convection equation with source term.

$$A\frac{d\phi}{dx} = cos(x) \tag{61}$$

2- 1D advective equation

$$\frac{d\phi}{dt} + \frac{d\phi}{dx} = 0 \tag{62}$$

3- 1D and 2D Euler equations in liner form

$$A\frac{\partial Q}{\partial x} + B\frac{\partial Q}{\partial y} = 0 \tag{63}$$

4- 2D Navier - Stokes equations in liner form

$$A\frac{\partial Q}{\partial x} + B\frac{\partial Q}{\partial y} = A_v\frac{\partial Q}{\partial x} + B_v\frac{\partial Q}{\partial y} \tag{64}$$

Where A, B, A_v and B_v are the Jacobian matrix and can be found in the literature (see, for example, reference [7]).

5- 1D convection-diffusion equation

$$A\frac{d\phi}{dx} - D\frac{d^2\phi}{dx^2} = 0 \tag{65}$$

Here details of the solution of these equations due to the length of paper and be less important are not presented and only the results are shown. Note that all examples are RFEM (the results for the NFFEM is almost identical). I used the infinite elements for non-solid boundaries in all examples were solved. I used the Gauss elimination method to solve the system of equations and also did not use any of the stabilizer term

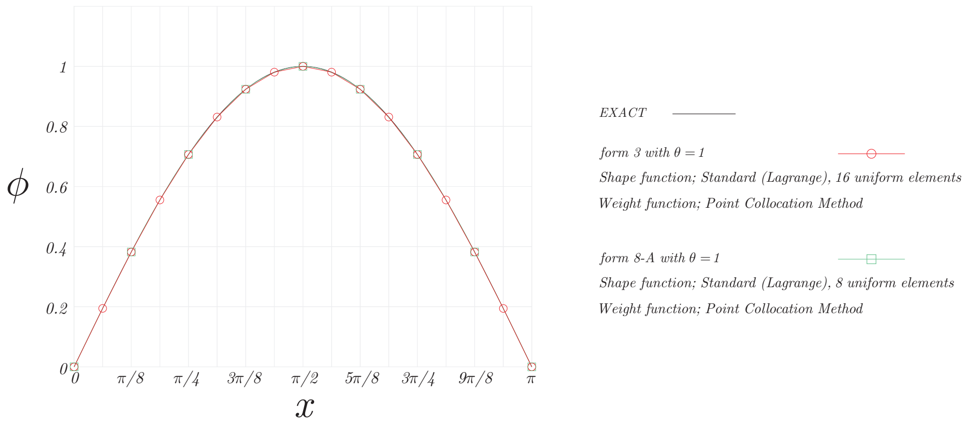

Figure 11: 1D convection equation with source term on $[0, \pi]$ and $A = 1$.

Figure 12: 1D advective equation.

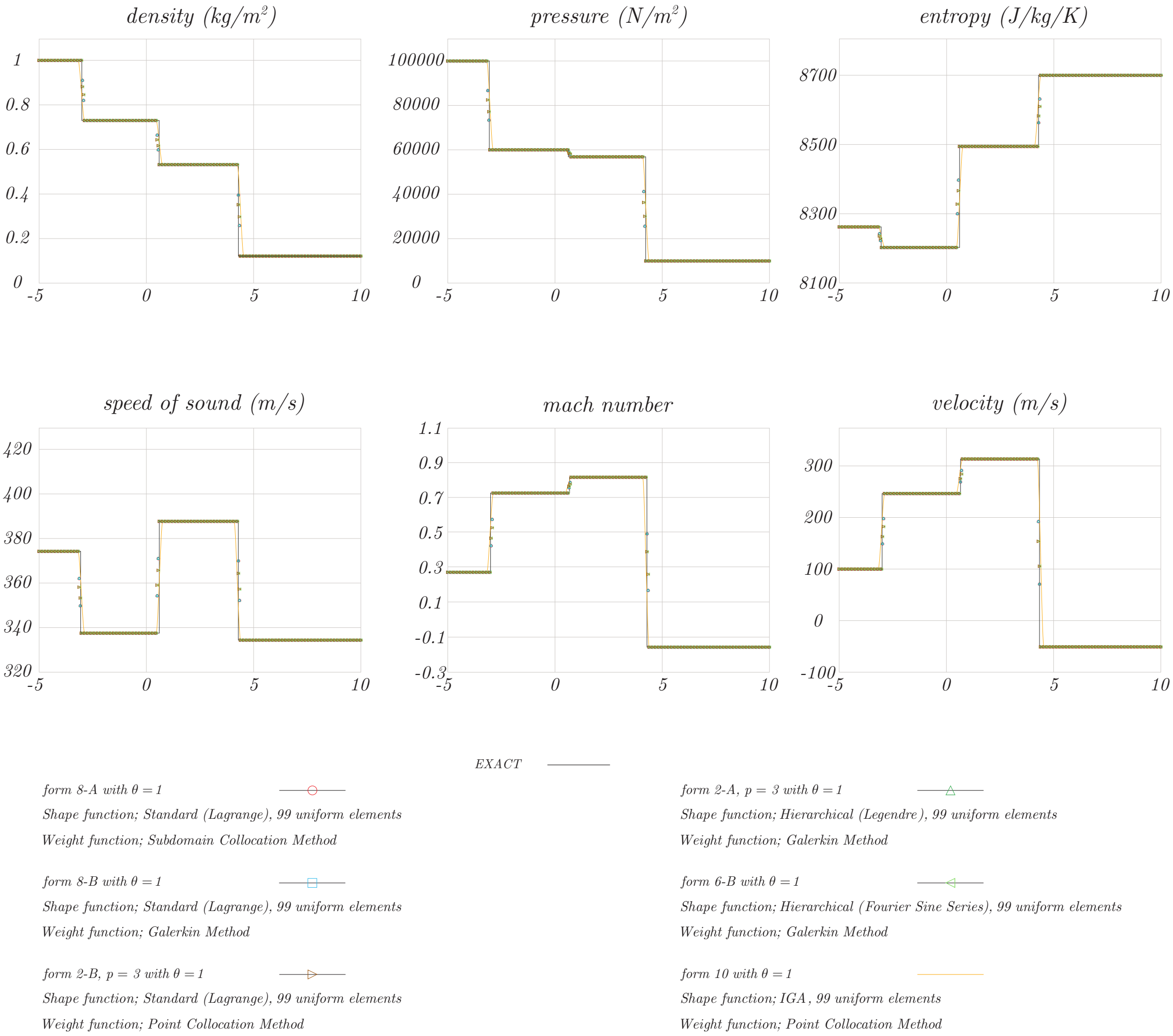

Figure 13: Roe`s approximate solution to the Riemann problem for the Euler equations 1D.

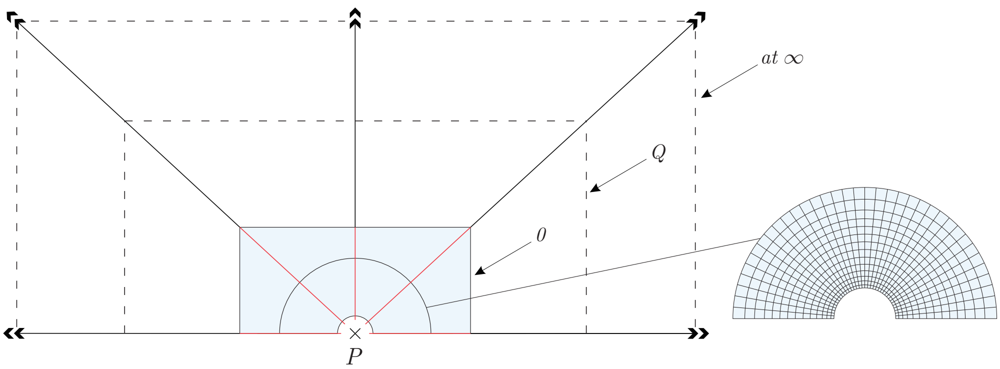

Figure 14: Grid for flow around a cylinder.

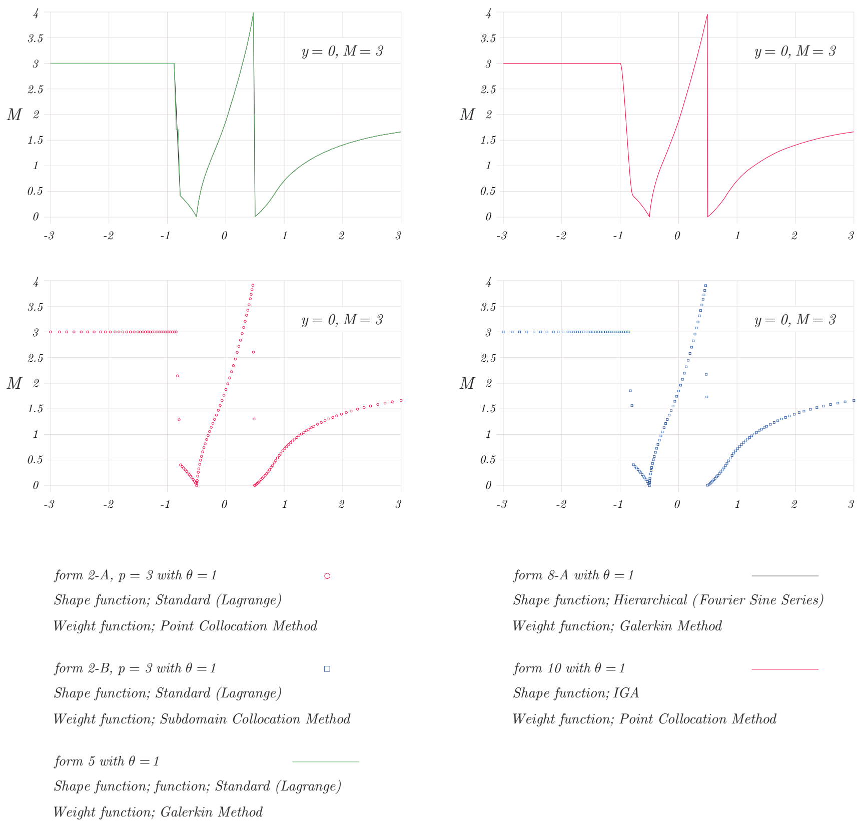

Figure 15: Flow around a cylinder (steady-state Euler equations).

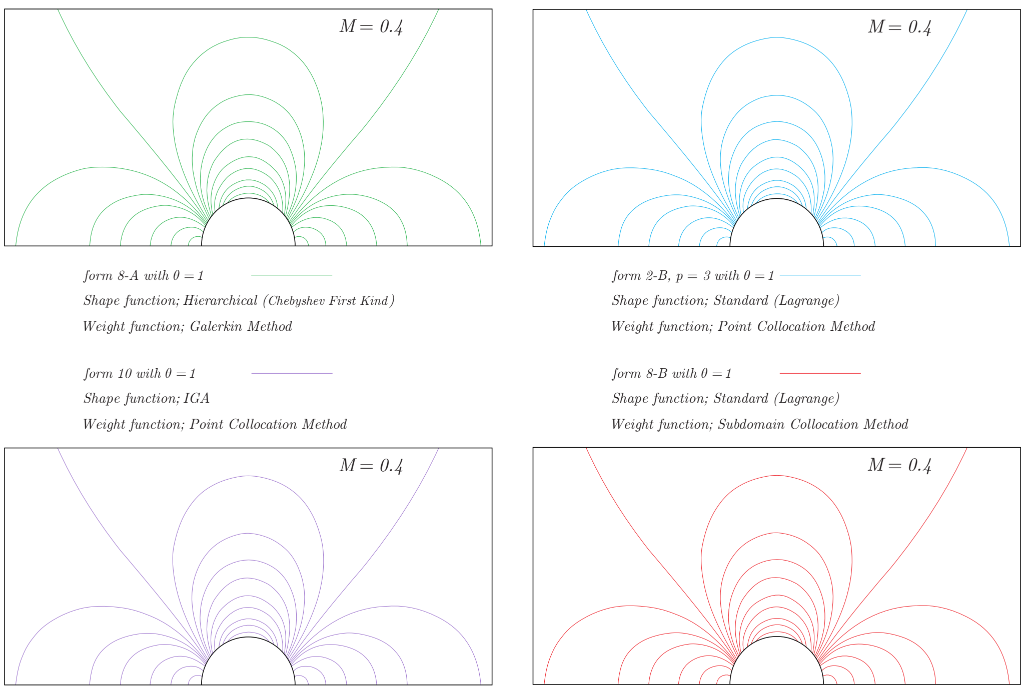

Figure 16: Mach contours for flow around a cylinder (steady-state Euler equations).

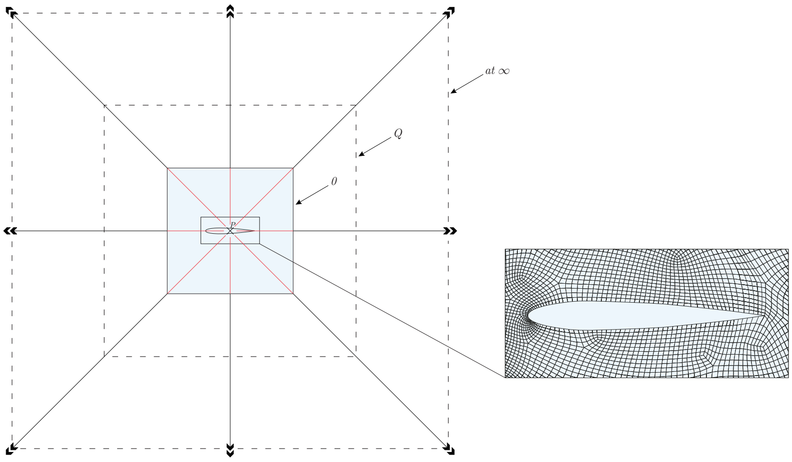

Figure 17: Grid for flow over a NACA 0012 airfoil.

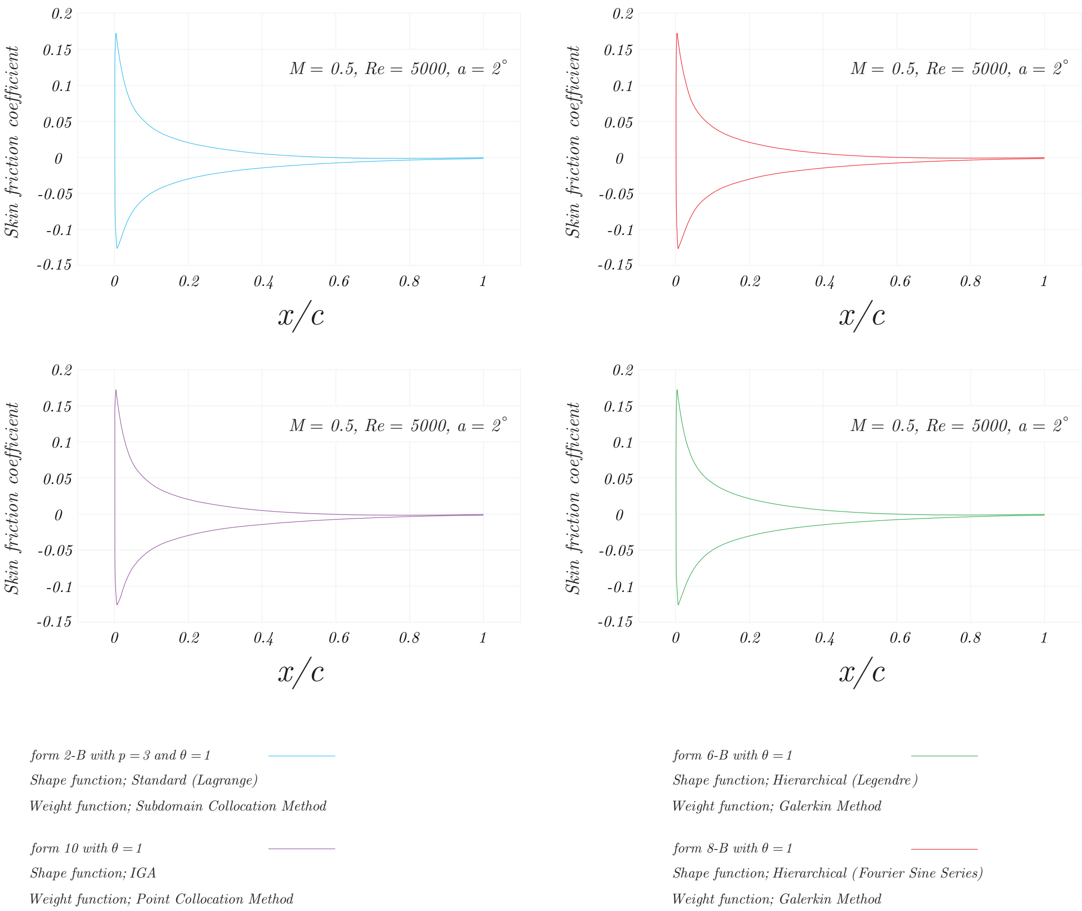

Figure 18. Skin friction distributions for laminar flow over a NACA 0012 airfoil (steady-state Navier-Stokes equations).

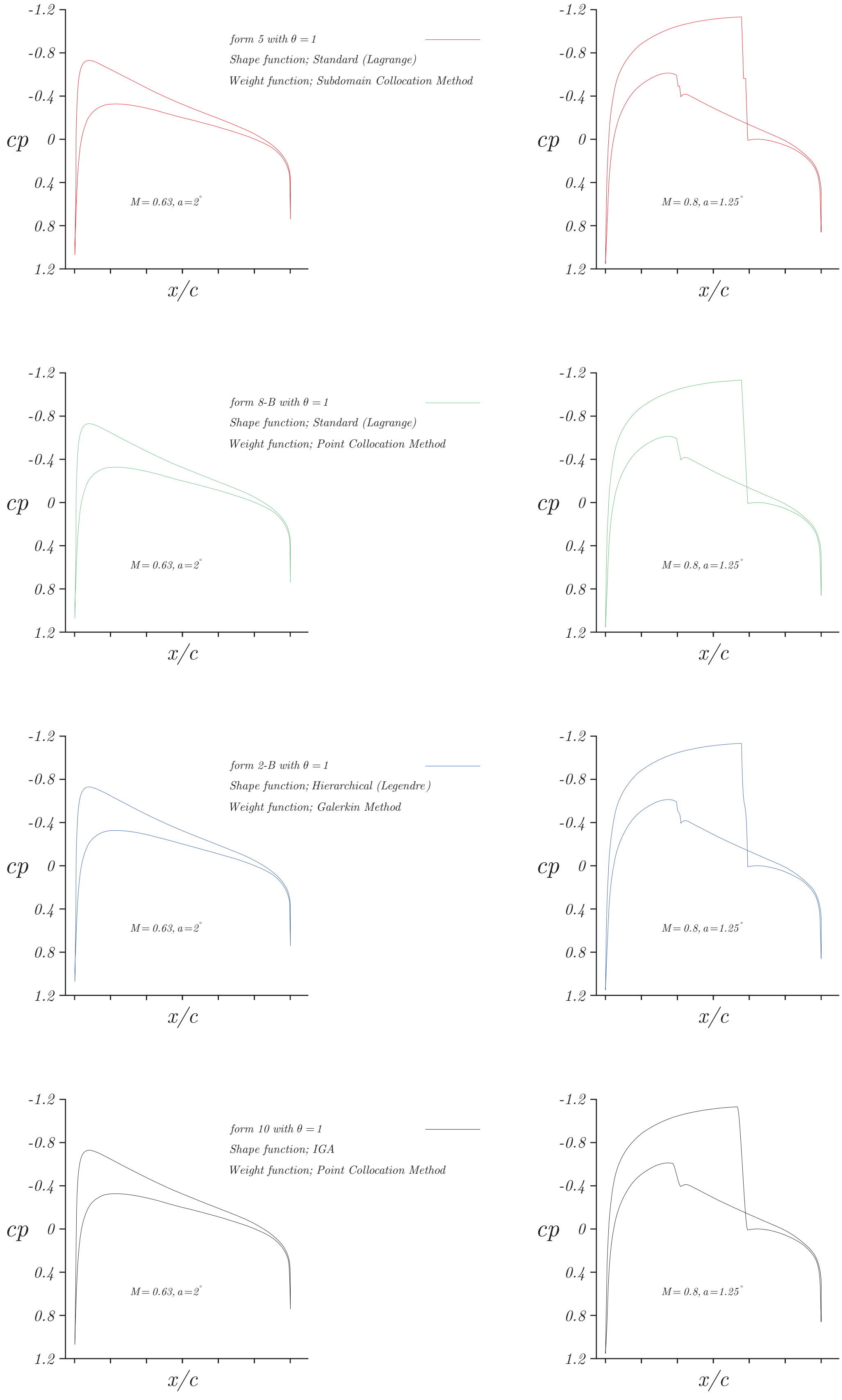

Figure 19: Surface pressure distribution for subcritical NACA 0012 airfoil (steady-state Euler equations).

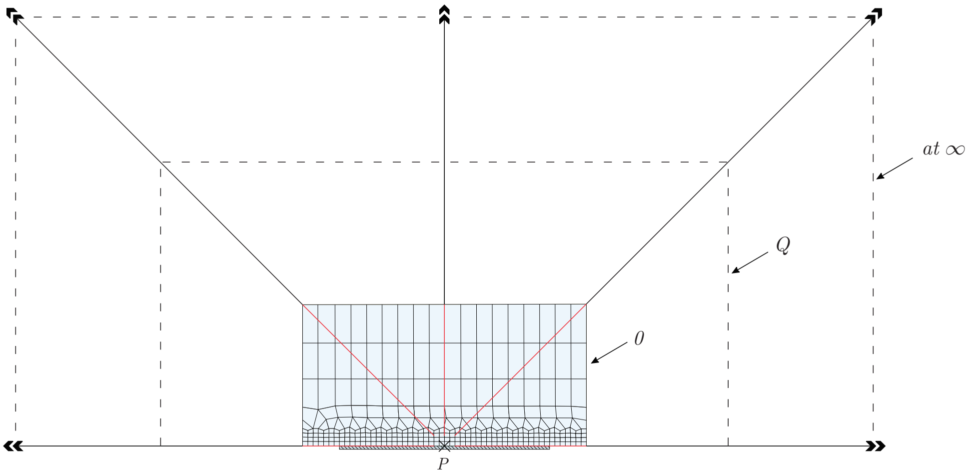

Figure 20: Solution domain and lines grid for flat plate flow.

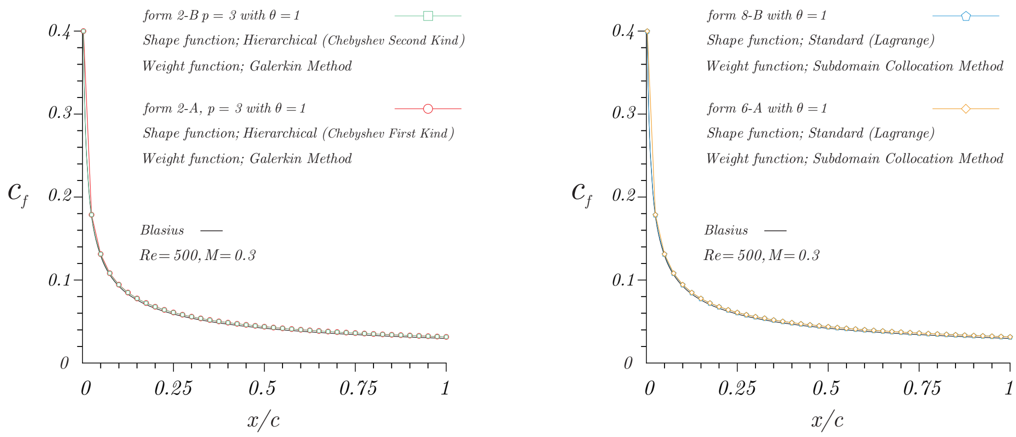

Figure 21: Skin friction distributions for flat plate flow (steady-state Navier-Stokes equations).

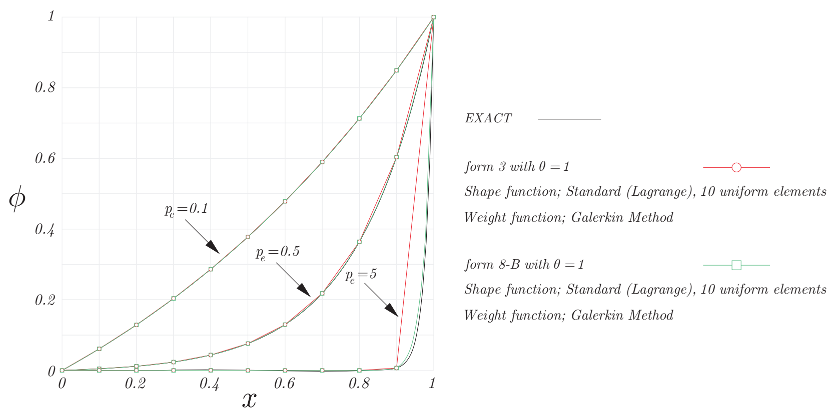

Figure 22: 1D convection-diffusion equation.

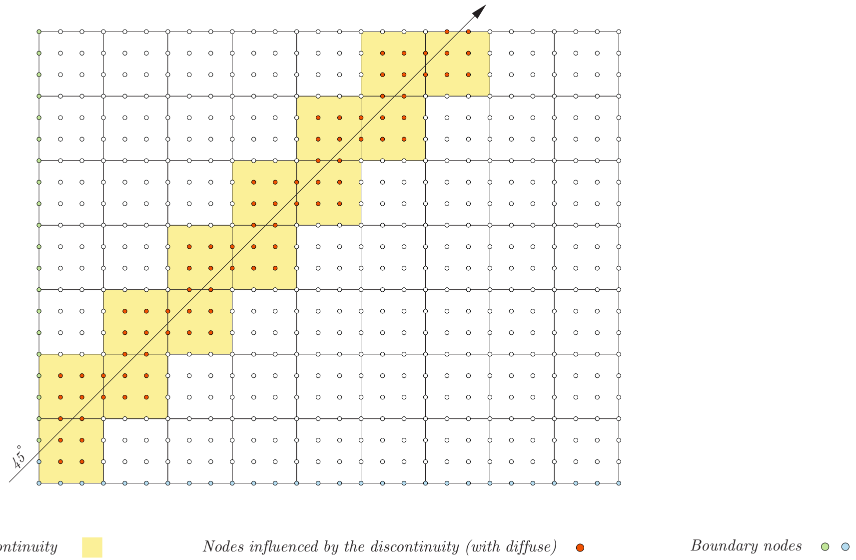

Figure 23: Elements and nodes influenced by the discontinuity when is used α_i' rather than α_i.

8 Conclusions

The best criterion for evaluate a numerical method is the results of the method and as can be seen in the examples were solved, the result of RFEM comparable to the best results were obtained by other methods is (for these examples, because of the small size of the elements, the difference between the results from different methods can not be seen) and given that the system of equations resulting from this approach similar (in terms of density) Finite Difference Method (FDM) is its efficiency can be compared with finite difference methods (although the use of the Legendre shape functions gives an efficiency much higher than FDM) so do not think other methods that are used in CFD be able to compete with it. Also, due to the similarity of relations many of the techniques used in the FDM can be used to RFEM [8].

References

[1] Khaleghi, M. R. A. A new computational package for using in CFD and other problems "European Journal of Natural History" №4 2015, pp. 11-32.
[2] Khaleghi, M. R. A. A very inexpensive scheme on RFEM to use in CFD and other problems "European Journal of Natural History" №4 2015, pp. 3-5.
[3] Zienkiewicz, O. C. and Morgan, K. Finite Elements and Approximation, John Wiley & Sons, 1983.
[4] Zienkiewicz, O. C. and Taylor, R. L. The Finite Element Method: Volume 1, the Basis, Butterworth-Heinemann, 2001.
[5] Hughes, T.J.R, Cottrell, J.A, Bazilevs, Y. "Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement". Computer Methods in Applied Mechanics and Engineering, 194 (39-41), pp. 4135–4195, 2005.
[6] Deng, J, Chen, F, Li, X, Hu, C, Tong, W, Yang, Z, and Feng, Y, Polynomial splines over hierarchical T-meshes. Graph. Models 70(4):76–86, 2008.
[7] Laney, C.B. Computational Gasdynamics. Publisher: Cambridge University Press. Pub. Date: June 28, 1998.
[8] Hoffmann, K.A. and Chiang, S.T. Computational Fluid Dynamics, Vol. I and II, 3rd edition, Engineering Education System (1998).